A rotation by $\varphi \in [0, 2\pi]$ about the z-axis of a vector $\mathbf{v} \in \mathbb{R}^3$ can be written

$$R_z(\varphi) = \begin{pmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{pmatrix}$$

which for infinitesimal $\varphi$ can be expanded as

$$R_z(\varphi) \approx \mathbf{1} - i \varphi \mathbf{T}_z$$

with $\mathbf{T}_z = \begin{pmatrix}0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}$ i.e. $R_z(\varphi) \approx \begin{pmatrix}1 & -\varphi & 0 \\ \varphi & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}$

(similarly for $R_x$ & $R_y$

with $\mathbf{T}_x = \begin{pmatrix}1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -1\end{pmatrix}$ $\mathbf{T}_y = \begin{pmatrix}1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1\end{pmatrix}$

(note that the matrix exp is given by $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \mathbf{1} + A + \frac{A^2}{2!} + \cdots$)

thus $R_z(\varphi) \approx \exp(-i \varphi \mathbf{T}_z)$

Now prod of two rotations like $e^{-i \theta \mathbf{T}_y} \cdot e^{-i \varphi \mathbf{T}_z}$ can always be written as a single exp $e^{-i \mathbf{\alpha} \cdot \mathbf{T}}$ where $\mathbf{\alpha} = (\alpha_x, \alpha_y, \alpha_z) = (0, \theta, \varphi)$

$\mathbf{T} = (T_x, T_y, T_z) = (0, T_y, T_z)$

More generally, suppose we’re interested in a prod of rot of the form

$$e^{-i \mathbf{\alpha} \cdot \mathbf{T}} \cdot e^{-i \mathbf{\beta} \cdot \mathbf{T}}$$

which we set $= e^{-i \mathbf{\alpha} \cdot \mathbf{T}}$

We wish to calculate $\mathbf{T}$ given $\mathbf{\alpha}$ & $\mathbf{\beta}$. Expanding the exp

$$= \left( \mathbf{1} - i \mathbf{\alpha} \cdot \mathbf{T} - \frac{1}{2} (\mathbf{\alpha} \cdot \mathbf{T})^2 + \cdots \right) \left( \mathbf{1} - i \mathbf{\beta} \cdot \mathbf{T} - \frac{1}{2} (\mathbf{\beta} \cdot \mathbf{T})^2 + \cdots \right)$$

$$= \left( \mathbf{1} - i (\mathbf{\alpha} + \mathbf{\beta}) \cdot \mathbf{T} - \frac{1}{2} ( (\mathbf{\alpha} + \mathbf{\beta}) \cdot \mathbf{T})^2 - \frac{1}{2} \left[ \mathbf{\alpha} \cdot \mathbf{T}, \mathbf{\beta} \cdot \mathbf{T} \right] + \cdots \right)$$

where $[\mathbf{A}, \mathbf{B}] = \mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{A}$

not necessarily 0 since rot don’t always commute
which can be rewritten as a single exp

\[ \exp \left[ -i (\vec{a} + \vec{b}) \cdot \vec{T} - \frac{1}{2} [\vec{a} \cdot \vec{T}, \vec{b} \cdot \vec{T}] + \ldots \right] \]

Thus, to this order in the exp, in order to calculate \( \vec{\gamma} \), we need to know the value of commutators such as \( [T_x, T_y] \) rather than ordinary prod like \( T_x T_y \). This is in fact true to all orders.

By direct computation, one finds

\[ [T_x, T_y] = i T_z , \quad [T_y, T_z] = i T_x , \quad [T_z, T_x] = i T_y \]

(These commutation relations, which we obtained by considering geometrical rotations, can now be used to form an Algebra known as a Lie Algebra).

**Def of an algebra**

Let \( A \) be a vs over a field \( F \), equipped with an additional binary operation \( \circ : V^2 \to V \). Then \( A \) is an algebra over \( F \) if the following hold for all \( \vec{x}, \vec{y}, \vec{z} \in A \) and all scalar \( a, b \in F \):

1. **right distributivity**: \( (\vec{x} + \vec{y}) \circ \vec{z} = \vec{x} \circ \vec{z} + \vec{y} \circ \vec{z} \)
2. **left dist**: \( \vec{x} \circ (\vec{y} + \vec{z}) = \vec{x} \circ \vec{y} + \vec{x} \circ \vec{z} \)
3. **scalar compatibility**: \( (a \vec{x}) \circ (b \vec{y}) = (ab) (\vec{x} \circ \vec{y}) \)

**Example**: \( A = \mathbb{R}^3 \), vector +, \( F = \mathbb{R} \), scalar prod

\( \circ = \) is vector cross prod \( \times \)

next, we take \( \circ \) to be \([x, y] \)

\( \) Lie bracket
So suppose there are three quantities $t_x$, $t_y$ and $t_z$ with a *Lie product* indicated by $[,]$ such that

\[
[t_x, t_y] = i t_z \quad [t_y, t_z] = i t_x \quad [t_z, t_x] = i t_y
\]  

(*)

We consider linear combinations of the $t_i$'s and further require that:

(i) *linearity* $[a \cdot t + b \cdot t, c \cdot t] = [a \cdot t, c \cdot t] + [b \cdot t, c \cdot t]$

(ii) *anti-symmetry* $[a \cdot t, b \cdot t] = -[b \cdot t, a \cdot t]$

For now, the prod $[a \cdot t, b \cdot t]$ is left unspecified (i.e. not a commutator). Once we represent the algebra by matrices, then the Lie product will have a well defined meaning.

The abstract Lie Algebra derived above from the rotation group displays the features which define Lie algebras in general.

A Lie algebra is a vector space $L$ (e.g. the linear combinations of the $t_i$'s) together with a bilinear op $[x,y] : L^2 \rightarrow L$ satisfying:

\[
\begin{align*}
[x_i + x_j, y] &= [x_i, y] + [x_j, y] \\
[a x_i, y] &= a [x_i, y] \\
[x, y] &= -[y, x]
\end{align*}
\]

We now seek the representations of the Lie Algebra defined by (*) i.e. a set of linear transformations (matrices) $T_x, T_y, T_z$ with the same commutation relations as the $t_i$'s.

(i.e. provide a basis for the algebra)
It is convenient to define

\[ t_+ = b_x + i b_y \quad t_- = b_x - i b_y \]

so that the commutation relation become

\[ [t_z, t_+] = t_+ \quad [t_z, t_-] = -t_- \quad [t_+, t_-] = 2 t_z \]

We now suppose that the \( t \)'s are to be rep by some linear transf:

\[ b_x \rightarrow T_x, \quad b_y \rightarrow T_y, \quad t_z \rightarrow T_z \]

where the \( T \)'s act on some vector space \( V \). We can in fact construct this space directly. Start with a single \( v_j \in V \) and define the actions of \( T_z \) & \( T_+ \) on it by

\[ T_z v_j = j v_j \quad T_+ v_j = 0 \]

Note: an eigenvector of a linear transformation is a non-zero vector whose direction does not change under that transformation:

\[ T \bar{v} = \lambda \bar{v}, \quad \lambda \in \mathbb{C} \]

Now consider \( T_- v_j \); and apply \( T_z \):

\[ T_z (T_- v_j) = (T_- T_z - T_z) v_j = T_- (j v_j) - T_z v_j \]

Thus \( T_- v_j \) is an eigenvector of \( T_z \) with eigenvalue \( j^{-1} \).

So let \( v_{j-1} = T_- v_j \). We can define additional vectors of \( V \) sequentially:

\[ v_{k-1} = T_- v_k \]
It is convenient to define
\[ t_+ = b_x + i b_y \quad t_- = b_x - i b_y \]
so that the commutation relation become
\[ [t_z, t_+] = t_+ \quad [t_z, t_-] = -t_- \quad [t_+, t_-] = 2 t_z \]

We now suppose that the \( t \)'s are to be rep by some linear transf:
\[ t_x \to T_x, \quad t_y \to T_y, \quad t_z \to T_z \]
where the \( T \)'s act on some vector space \( V \). We can in fact construct this space directly. Start with a single \( v_j \in V \) and define the actions of \( T_z \) & \( T_+ \) on it by
\[ T_z v_j = j v_j \quad T_+ v_j = 0 \]

Note: an eigenvector of a linear transformation is a non-zero vector whose direction does not change under that transformation:
\[ T \vec{v} = \lambda \vec{v}, \quad \lambda \in \mathbb{C} \]

Now consider \( T_- v_j \); and apply \( T_z \):
\[ T_z (T_- v_j) = (T_- T_z - T_z) v_j = T_-(j v_j) - T_+ v_j \]
\[ [t_z, t_-] = -t_- = (j-1) T_- v_j \]
Thus \( T_- v_j \) is an eigenvector of \( T_z \) with eigenvalue \( j-1 \).
So let \( v_{j-1} = T_- v_j \). We can define additional vectors of \( V \) sequentially:
\[ v_{k-1} = T_- v_k \]
Now if our space \( V \), which consists of all linear combinations of the \( V \)'s is to be finite dimensional, this procedure must terminate somewhere for some \( q \) :

\[
T_\text{-} Vq = 0.
\]

In order to determine \( q \), we must consider the action of \( T_+ \).

Note that

\[
T_\text{z} \left( \frac{T_+ V_k}{T_+ T_\text{z} V_k} \right) = T_+ V_k + T_+ k V_k = T_+ (k+1) V_k
\]

\[
[i_\text{z}, i_\text{z}] = 0
\]

i.e. \( T_+ V_k \) is an eigenvector of \( T_\text{z} \) eigenvalue \( k+1 \).

Can show by induction that \( T_+ V_k \propto V_{k+1} \), \( \{ \text{i.e. } T_+ V_0 \propto V_1 \}

T_+ V_n \propto V_{n+1} \Rightarrow T_+ V_{n+1} \propto V_{n+2}

So let \( r_k \) be that const. of proportionality

\[
r_k V_{k+1} = T_+ V_k
\]

\[
= T_+ T_- V_{k+1} \quad \Leftarrow V_{k+1} \propto T_- V_k
\]

\[
= (T_- T_+ + 2T_\text{z}) V_{k+1} \quad \Leftarrow [T_\text{z}, T_-] = 2k_\text{z}
\]

\[
= T_- T_+ V_{k+1} + 2T_\text{z} V_{k+1}
\]

\[
= T_- r_{k+1} V_{k+2} + 2(k+1) V_{k+1}
\]

\[
= r_{k+1} V_{k+1} + 2(k+1) V_{k+1}
\]

\[
= [r_{k+1}, 2(k+1)] V_{k+1}
\]

(This recursion relation is easy to satisfy. (Since \( T_+ V_j \equiv 0 \) \( \text{(x)} \)

we use the condition \( r_j = 0 \), ) we have

\[
r_{k+1} - r_k = -2(k+1)
\]

General soln: \( r_k = c_1 - k(k+1) \)

with \( \text{(x)} \) \( r_j = c_1 - (j+1)(j+2) = 0 \Rightarrow c_1 = \tfrac{1}{2}(j+1) \\ \Rightarrow r_k = \tfrac{1}{2}(j+1) - k(k+1) \)
Now we can find the value of $q$ as defined by $T_+ \cdot T_- v_q = 0$:

\[
T_+ T_- v_q = 0 \\
= (T_- T_+ + 2T_z) v_q \\
= [\frac{j(j+1)}{2} - q(q+1)] v_q + 2T_z v_q \\
= [\quad (""") \quad] v_q + 2q v_q
\]

so $\frac{j(j+1)}{2} - q(q+1) + 2q = 0$

roots: $q = j+1$ however, should have $q \leq j$ since $j$ was our starting point (upper bound)

$q = -j$

Thus $q = -j$.

(\text{In this way we have recovered the familiar rep of the rotation grp, or more accurately, of its Lie algebra.})

* The eigenvalues of $T_z$ range from $j \to -j$. The $2j + 1$-dim rep constructed above is said to be irreducible, i.e. there is no proper subspace of $V$ which is mapped into itself by the various $T$'s.

* The rep of smallest dim has $j = \frac{1}{2}$. Its matrices $T$ are $2 \times 2$ & traceless:

\[
\{ (0, 0), (0, -i), (-i, 0) \}
\]

The matrices $T_x$, $T_y$ and $T_z$ are hermitian, linear combs of which yield the $T$'s, traceless & hermitian.

* The matrices $\exp(iT)$ form a group of unitary matrices of unit determinant... $SU(2)$, for special unitary.