The unitary groups can be defined in terms of quantities which are left invariant. Consider a general complex transformation \( \mathbf{x}' = A \mathbf{x} \)

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

where \( a, b, c, d \in \mathbb{C} \), so there are 2x4 free parameters, and we assume \( ad - bc \neq 0 \).

**Unitary Transformations**

Suppose we require the norm \( \| \mathbf{x} \|^2 = (x)^2 + (y)^2 \) to be an invariant of \( A \).

Then \( \| \mathbf{x}' \|^2 = \| A \mathbf{x} \|^2 = \| \mathbf{x} \|^2 \) requires

\[
\begin{align*}
(a^* a + b^* b) x^2 + (c^* c + d^* d) y^2 + (2a^* b + 2c^* d) xy \\
&= (|a|^2 + |b|^2) x^2 + (|c|^2 + |d|^2) y^2 + (2\text{Re}(a^* b) + 2\text{Re}(c^* d)) xy
\end{align*}
\]

now \( \| \mathbf{x}' \|^2 = \| \mathbf{x} \|^2 \) requires

\[
\begin{cases}
|a|^2 + |b|^2 = 1 \\
|c|^2 + |d|^2 = 1 \\
ab + cd^* = 0 \quad (\rightarrow a^* b + c^* d = 0)
\end{cases}
\]

Four conditions total \( (2 + \text{Re} + \text{Im}) \) down from 8 parameters.

These amount to requiring \( A^* A = 1 \). The group comprised of unitary 2x2 matrices is denoted \( U(2) \). If in addition we require unit determinant

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad |a|^2 + |b|^2 = 1
\]

(1)

we are left with three conditions. The group of these matrices is denoted by \( SU(2) \).

If the matrix is of the general unitary matrix (1) are expressed in terms of their real & complex parts, can decompose it into the form of a "basis":

\[
U = \begin{pmatrix} a_r + i a_i; & b_r + i b_i; \\ -b_r + i b_i; & a_r - i a_i; \end{pmatrix} = a_r \begin{pmatrix} 1 & i a_i \\ -i a_i & 1 \end{pmatrix} + i a_i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b_r \begin{pmatrix} 0 & i b_i \\ -i b_i & 0 \end{pmatrix} + i b_i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i b_i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
Thus any $2 \times 2$ unitary matrix can be represented as a linear combination of the unit matrix and the matrices

\[
\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

where $\sigma_i^2 = 1$ and $\sigma_i \sigma_j = i \epsilon_{ijk} \sigma_k$. 
In the Lagrangian formulation, the fundamental law of motion is the Euler-Lagrange equation:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}$$

for a classical particle

$$\frac{\partial L}{\partial q_i} = \frac{\partial T}{\partial \dot{q}_i} = m \ddot{x}_i \quad \text{"leads to Newton's 2nd law for conservative forces"} \quad \frac{\partial L}{\partial \dot{q}_i} = -\frac{\partial U}{\partial \dot{x}_i}$$

while a particle is localized, a field occupies some region of space

$$x(t), y(t), z(t) \quad \text{vs.} \quad \phi_i (x, y, z, t)$$

scalar or vector field

In field theory, we consider a Lagrangian density $\mathcal{L}$, which is a function of the fields $\phi_i$ and their $x, y, z, t$ derivatives:

$$\partial_m \phi_i = \frac{\partial \phi_i}{\partial x^m}$$

In the relativistic case, the E-L equations generalize to

$$\partial^\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \right) = \frac{\partial \mathcal{L}}{\partial \phi_i} \quad i = 1, 2, 3, 4$$

Example: Klein-Gordon Lagrangian for a Scalar (spin-0) field

Suppose we have a single scalar field $\phi$

$$\mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi \right) \left( \partial^\mu \phi \right) - \frac{1}{2} \left( \frac{mc}{\hbar} \right)^2 \phi^4$$

in this case

$$\partial_\mu \phi \partial^\mu \phi - \partial_\mu \phi \partial_\mu \phi - \ldots$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = \partial^\mu \phi$$
\[ \frac{\partial L}{\partial \phi} = -\left( \frac{mc}{k} \right) \phi \]

hence the E-L formula yields
\[ \partial_{\mu} \partial^{\mu} \phi + \left( \frac{mc}{k} \right) \phi = 0 \]

the KG eqn describing a spin 0 particle of mass \( m \).

- **Other ex:** Dirac Lagrangian for a spinor (spin 1/2) field \( \psi \):
  \[ L = i \left( \frac{ic}{k} \right) \bar{\psi} \gamma^{\mu} \partial_{\mu} \psi - (mc') \bar{\psi} \psi \]
  \[ \rightarrow i \gamma^{\mu} \partial_{\mu} \psi - \left( \frac{mc}{k} \right) \psi = 0 \]
  the Dirac eqn, describing a spin 1/2 particle of mass \( m \).

- Proca Lagrangian for a vector (spin 1) field \( A^{\mu} \):
  \[ L = \frac{1}{16\pi} \left( \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \right) \left( \partial_{\nu} A^{\mu} - \partial_{\mu} A^{\nu} \right) + \frac{i}{2\pi} \left( \frac{mc}{k} \right) A^{\nu} A_{\nu} \]
  \[ \rightarrow \partial_{\mu} F^{\mu} - \left( \frac{mc}{k} \right) A^{\nu} A_{\nu} = 0 \text{, where } F^{\mu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \]

note that with \( m=0 \), we are left with Maxwell's eqn for empty space.

"Note further that these \( L \) are not unique: one can multiply \( L \) by a constant, add a constant, or the divergence of an arbitrary vector function—these will cancel out in the E-L eqn."

--
Notice that the Dirac Lagrangian
\[ \mathcal{L} = i \hbar c \bar{\psi} \gamma^\mu \partial_\mu \psi - mc^2 \bar{\psi} \psi \]
is invariant under the transformation
\[ \psi \rightarrow e^{i \theta} \psi \] (global phase transformation)
where \( \theta \in \mathbb{R} \); since \( \bar{\psi} \) then becomes \( e^{-i \theta} \bar{\psi} \).

so that the exp cancel out in both terms of the \( \mathcal{L} \).

However, if we let the phase vary with space & time, i.e. \( \theta = \theta(x^\mu) \)
\[ \psi \rightarrow e^{i \theta(x)} \psi \] (local phase transf.)
the \( \mathcal{L} \) picks up an extra term from the derivative of \( \theta \)
\[ \partial_\mu (e^{i \theta} \psi) = i (\partial_\mu \theta) e^{i \theta} \psi + e^{i \theta} \partial_\mu \psi \]
so that
\[ \mathcal{L} \rightarrow \mathcal{L} - \hbar c (\partial_\mu \theta) \bar{\psi} \gamma^\mu \psi \]
Leibniz \( \lambda(x) \equiv -\frac{\hbar c}{q} \theta(x) \), \( q \) being the charge of the particle involved
this becomes
\[ \mathcal{L} \rightarrow \mathcal{L} + \left( q \bar{\psi} \gamma^\mu \psi \right) \partial_\mu \lambda \]
under the local phase transformation \( \psi \rightarrow e^{-i \frac{q}{\hbar c} \lambda(x)} \psi \)

• What if we require the \( \mathcal{L} \) to be invariant under local phase transformations?
Suppose
\[ \mathcal{L} = \left( i \hbar c \bar{\psi} \gamma^\mu \partial_\mu \psi - mc^2 \bar{\psi} \psi \right) - (q \bar{\psi} \gamma^\mu \psi) A_\mu \]
where the new field \( A_\mu \) transforms as
\[ A_\mu \rightarrow A_\mu + \partial_\mu \lambda \]
This "new" $\mathcal{L}$ is now locally invariant, at the cost of a new vector field that couples to $\psi$. However, the complete $\mathcal{L}$ must then include a 'free' term for the field $A^\mu$ itself:

"since it's a vector, we turn to the Proca $\mathcal{L}$:

$$\mathcal{L} = \frac{-i}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{1}{18\pi} \left( \frac{m_A}{\hbar} \right) A^\mu A^\mu$$

However, while $F^{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu)$ is invariant under (x)

$A^\mu A^\nu$ isn't. Thus, the new field must be massless ($m_A = 0$)
or invariance will be lost.

"If we start with the Dirac $\mathcal{L}$, and demand local phase invariance we are forced to introduce a massless vector field so that:

$$\mathcal{L} = \left( i \hbar c \gamma^\mu \partial_\mu \psi - m c^2 \psi \right) - \left( \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \right) + \left( 9 \bar{\psi} \gamma^\mu \psi \right) A_\mu$$

free term for $A_\mu$

Maxwell $\mathcal{L}$ with $J_\mu = e q \bar{\psi} \gamma^\mu \psi$

How did we obtain the extra term ($\psi \psi$)?

→ The cliff between global & local phase transf arises when we calculate

$$\partial_\mu (e^{i\theta} \psi) = i (\partial_\mu \theta) e^{i\theta} \psi + e^{i\theta} \psi$$

$$\Rightarrow \partial_\mu \psi \rightarrow e^{-i\lambda/\hbar c} \left[ \partial_\mu - i \frac{q}{\hbar c} (\partial_\mu \lambda) \right] \psi \quad (\dagger)$$

so that we pick up an extra $\partial_\mu \lambda$. If in the original $\mathcal{L}$, we replace every deriv $(\partial_\mu)$ by the covariant derive

$$\partial_\mu \equiv \partial_\mu + i \frac{q}{\hbar c} A_\mu \quad (\lambda \rightarrow D_\mu) \quad (\text{cf. } \nabla^\mu \psi = \partial_\mu \psi + \frac{\gamma_\mu}{\gamma^\nu} \partial_\nu \psi)$$

the gauge transf of $A_\mu$ will cancel the extra term in $(\dagger)$:

$$\partial_\mu \psi \rightarrow e^{-i\lambda/\hbar c} \partial_\mu \psi$$

and the invariance of $\mathcal{L}$ is restored.
This substitution, converting a globally inv $L$ into a locally inv one
is called the minimal coupling rule. Our starting point was to
consider global phase transf., i.e. mult by $1 \times 1$ unitary matrices

$$\psi \rightarrow U \psi, \quad U^* U = 1$$

"The group of all such mat is $U(1)$ - hence the symmetry involved
is called $U(1)$ gauge invariance."

Next we look at $U \in SU(2)$. 