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A solution to the hold-up problem involving gradual investment

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Abstract

We consider a setting in which the buyer's ability to hold up a seller's investment is so severe that there is no investment in equilibrium of the static game typically analyzed. We show that there exists an equilibrium of a related dynamic game generating positive investment. The seller makes a sequence of gradually smaller investments, each repaid by the buyer under the threat of losing further seller investment. As modeled frictions converge to zero, the equilibrium outcome converges to the first best. We draw connections between our work and the growing literature on gradualism in public good contribution games and bargaining games.

JEL classification: D23; C73; L14

Keywords: Hold-up problem; Gradualism; Incomplete contracts; Investment; Contribution games

1. Introduction

A standard setting in which the hold-up problem arises involves investment by one party, call it the seller, which benefits another, call it the buyer, where this investment and its associated benefits cannot be verified by a court. Since it may be difficult to specify payment for the investment in a contract, the buyer may not have an incentive to compensate the seller fully; and, consequently, the seller will underinvest.

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To make our subsequent results as stark as possible, in this paper we consider an extreme form of the hold-up problem in which no contracts over investments or its benefits are possible and in which the buyer can appropriate all of the benefits from the seller's investment without providing any compensation. The hold-up problem is so severe in this setting that in the static game typically analyzed, which involves the buyer's paying the seller after the seller's investment is completed, there is no investment in equilibrium.

Interpreting "observability of investment" to mean that the buyer can observe the path of the seller's investment rather than just the aggregate amount, we show that the hold-up problem can be solved (or at least ameliorated) by moving from the static game typically analyzed to a dynamic game. In the dynamic game, the single lump of investment from the static game is divided into a sequence of installments, each consisting of an incremental investment by the seller followed by reimbursement by the buyer. The installments continue until the process breaks down due to exogenous frictions in the environment. Breakdown occurs with positive probability after each installment according to the outcome of a public randomizing device. We show that, for a broad set of parameters, there exists an equilibrium in the dynamic game generating positive investment. We show, further, that there exists an equilibrium of the dynamic game in which investment by the seller comes arbitrarily close to the first-best level as the probability of breakdown approaches zero. These results are striking recalling that the extreme form of the hold-up problem we have assumed yields no investment in the typical static game.

In the dynamic game, the buyer's incentive to pay the seller for each installment stems from the threat that the seller will not continue with further investment otherwise. A given installment is constrained not to be too large relative to future investment or else the buyer's benefit from deviation—its gain from not repaying a particular installment—would exceed the punishment—the loss of further investment. From this insight, one can draw several conclusions about the structure of the equilibrium investment sequence. First, to avoid unraveling, there cannot be a known, finite end to the number of installments. Second, as investment gradually accumulates toward its upper bound, the prospect of losing further investment becomes a less severe punishment, implying that the investment installments must gradually shrink to prevent the buyer from deviating.

The sequential investment equilibrium proposed in this paper as a solution to the hold-up problem shares features of strategies used in practice. The "indefinite delivery/indefinite quantity" strategy, used for billions of dollars of federal, state, and local government projects ranging from construction of passenger rail in Atlanta to renovation of affordable housing in Baltimore [13], is a staged procurement process that allows a party to end the process after each stage conditional on past experience, for example when a buyer decides a seller's quality has been unacceptably low. The literature on procurement management has suggested informally that this strategy, known variously as job order contracting (JOC), delivery order contracting (DOC), and simplified acquisition of base engineering requirements (SABER), may give sellers an incentive to provide high quality without resorting to detailed contracts [17].

Our paper is most closely related to the literature on gradual contributions to a public good [1,2,8,19], gradual concessions in bargaining [7], and subsequent generalizations [9,12,18]. The contribution of our paper on a conceptual level is to apply the idea of gradualism to a new context, the hold-up problem. Our result that equilibrium investment increments must gradually shrink follows the arguments in [7,9]. Our asymptotic efficiency result echoes the first efficiency result to appear in the related literature [2], and is closely related to the subsequent asymptotic efficiency results in [18,19].[1] A contribution of our paper on a formal level is to extend the results on asymptotic efficiency to the case of asymmetric players—the hold-up problem we study involves a seller which makes investments and a buyer which makes payments, so players are necessarily asymmetric in our context—whereas [18,19] focused on symmetric players with identical payoff functions.

Our paper is part of the literature proposing solutions to the hold-up problem in the presence of incomplete contracts, solutions ranging from asset ownership [14,16] to relational contracts [3] to implementation mechanisms [20]. A bargaining literature demonstrates that the hold-up problem may be less severe than in the standard model (e.g., [14]) if one adopts an alternative bargaining game or solution concept [6,10,15,23]. A growing literature in contract theory shows that, if contracts are assumed to exist but are partially incomplete in that they are signed prior to and are unconditional on the realization of certain relevant random variables, there are ex post renegotiation mechanisms and breach penalties that can still yield the first best. Of particular relevance in this literature is [5], which shows that sequencing investments, in the sense of having parties alternate their bilateral investments, can help solve the hold-up problem when investments have externalities.

There is a related corporate finance literature recognizing the benefits of staged investment, the most relevant being [22]. The author considers a venture capital project in which an entrepreneur may hold up the contribution of an investor by threatening to repudiate their contract after the investor sinks a capital investment. By sequencing investment, the investor is enabled to build up collateral, gradually improving its bargaining position and mitigating the hold-up problem. Since our model does not have collateral, the bargaining position of the party subject to hold up (seller in our case) does not improve over time as in [22], but if anything deteriorates as the amount of remaining investment declines and exerts less discipline on the buyer not to deviate from repayment. The second-best investment sequences in the two models have quite different structures: in [22], the optimum is a finite sequence with (eventually) increasing installments; in our paper, the optimum is a potentially infinite sequence of decreasing installments.

2. Model

First consider a benchmark, static model of the hold-up problem. There are two risk-neutral players, a seller and a buyer, and two periods, ex ante and ex post.

[1] Gale [12] provides general conditions for asymptotic efficiency.

Ex ante, the seller makes investment expenditure k, which is equivalently the seller's disutility of investment. Ex post, the buyer makes a payment p to the seller and consumes the benefit $u(k)$ from the seller's investment. Assume $u(k) \geqslant 0$, $u'(k) > 0$, and $u''(k) < 0$ for all $k \in [0, \infty)$. Assume there is no discounting within or between periods. Then net social surplus can be represented by $\phi(k) = u(k) - k$. Assume ϕ has an interior maximizer, $k^* \in (0, \infty)$. Note that k^* is the first best level of investment.

Following the literature on incomplete contracts, assume that k is observable to both players but neither k nor $u(k)$ is verifiable in court. To focus on the most extreme form of the hold-up problem, assume that the buyer has control rights over the benefits from the seller's investment. That is, the buyer cannot be excluded from enjoying the benefits of ex ante investment. This assumption is for expositional purposes only: with other control-rights regimes the hold-up problem, though less severe, would still exist; and so there would still be a role for the noncontractual mechanism we devise in this paper.

In this benchmark model, the buyer is only able to observe the seller's total investment k upon completion rather than the investment path. Because of the severity of the hold-up problem in our model, it is immediate that the seller's unique equilibrium choice is $k = 0$. Once the seller has invested, the buyer can appropriate the benefits without paying and so has no incentive to pay. Contractual incompleteness prevents them from signing contracts either forcing the seller to invest or the buyer to pay. Anticipating this, the seller has no incentive to invest.

For the remainder of the paper, we will study a dynamic extension of the benchmark model, allowing the seller to undertake a sequence of investments for which it receives a sequence of payments from the buyer. The goal of the subsequent analysis will be to determine whether positive investment can be sustained in equilibrium of this dynamic extension and whether social welfare can approach the first best under some conditions.

The ex ante investment stage is divided into subperiods indexed by $t \in \mathbb{N}$. The timing of a representative subperiod is given in Fig. 1. At the beginning of subperiod t, the seller makes incremental investment $\Delta k_t \geqslant 0$. Next, the buyer observes this investment and makes incremental payment $\Delta p_t \geqslant 0$ to the seller. The final step involves an exogenous friction. We have in mind the same sort of frictions that lead to the exogenous probability of breakdown of bargaining familiar from Binmore et al. [4]. In particular, the seller–buyer relationship ceases for the remainder of the ex ante stage with probability $\theta \in [0, 1]$ and continues into subperiod $t + 1$ with complementary probability, depending on the outcome of a public randomization.

There are several reasons for considering the public randomizations. On technical grounds, the public randomizations guarantee that the number of subperiods of investment and reimbursement is almost surely finite. On practical grounds, public randomizations are a way of capturing frictions that may arise in applications. Explicitly incorporating θ into the model allows us to analyze the comparative static effect of such exogenous frictions on the amount of investment that can be sustained and thus the remaining severity of the hold-up problem. It will be a simple exercise to take the limit as θ approaches zero to determine the equilibrium in the absence of

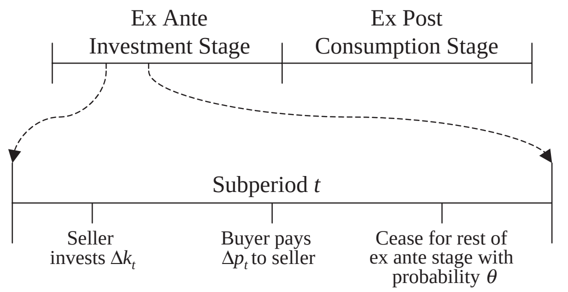

Fig. 1. Timing of game.

frictions. It should be emphasized that the equilibrium does not require the public randomizations for existence; indeed, higher values of θ will lead to lower equilibrium levels of investment and lower welfare.

Let $k_t = \sum_{i=1}^{t} \Delta k_t$ denote the cumulative investment from the beginning of the ex ante stage through subperiod t and $p_t = \sum_{i=1}^{t} \Delta p_t$ denote the cumulative payment. Let $K = \{k_t \mid t \in \mathbb{N}\}$ denote the entire sequence of equilibrium cumulative investments and $P = \{p_t \mid t \in \mathbb{N}\}$ denote the entire sequence of equilibrium cumulative payments.

Assume the seller's disutility from investing depends only on the total investment expenditure by the end of the ex ante stage k and not on the particular investment path. Likewise, the buyer's ex post benefit $u(k)$ depends only on total expenditure k and not on the particular investment path. Assume the buyer must wait until after the ex ante stage to consume the benefits from investment and that the seller cannot continue to make investments in the ex post stage.

The crucial assumption differentiating the dynamic extension from the benchmark, static model is on the nature of observability of investment. We assume investments and payments are potentially divisible into arbitrarily fine increments, with individual increments observable to both players. We further assume that these observations and payments are made without loss of social surplus. One can think of the seller as investing at a certain rate but pausing periodically to allow the buyer to observe whether certain milestones have been reached; in the extreme, the milestones may reflect arbitrarily small progress.

3. Extremal equilibrium

As is typical with dynamic games of the sort we are considering (see, e.g., [18] and of course the earlier work on the Folk Theorem, e.g., [11]), there may be multiple subgame perfect equilibria. In the present game, equilibria range from the least efficient, in which neither player participates and so there is no investment, to the extremal equilibrium, in which social surplus is maximized. In this section, we will solve explicitly for the extremal equilibrium. With an explicit solution in hand, it will be straightforward to prove our central results, namely that the extremal equilibrium involves positive investment and that investment approaches the first best as the friction approaches zero.

Let V_{Bt} be the expected equilibrium payoff for the buyer conditional on having reached subperiod t successfully (i.e., without a breakdown in the seller–buyer relationship). We have

$$V_{Bt} = \theta[u(k_t) - p_t] + (1-\theta)V_{Bt+1}, \tag{1}$$

implying

$$V_{Bt} = \sum_{i=0}^{\infty} \theta(1-\theta)^i [u(k_{t+i}) - p_{t+i}]. \tag{2}$$

The factor $\theta(1-\theta)^i$ in Eq. (2) is a hazard probability: the probability that the seller–buyer relationship breaks down at the end of subperiod $t+i$ conditional on having continued to subperiod t. If the seller–buyer relationship breaks down immediately after subperiod $t+i$, the buyer enjoys the benefit $u(k_{t+i})$ of the cumulative investment k_{t+i} having made cumulative payment p_{t+i} to the seller.

Analogously, let V_{St} be the expected equilibrium payoff for the seller conditional on having reached subperiod t successfully. We have

$$V_{St} = \theta(p_t - k_t) + (1-\theta)V_{St+1}, \tag{3}$$

implying

$$V_{St} = \sum_{i=0}^{\infty} \theta(1-\theta)^i (p_{t+i} - k_{t+i}). \tag{4}$$

To ensure there is no deviation from the subgame perfect equilibrium, the following incentive compatibility constraints must hold for all $t \in \mathbb{N}$: $V_{Bt} \geqslant V_{Bt}^{d}$ and $V_{St} \geqslant V_{St}^{d}$, where V_{Bt}^{d} and V_{St}^{d} are, respectively, the buyer's and seller's maximum deviation payoffs in subperiod t. The buyer can always deviate by "exiting" the ex ante stage, appropriating the seller's investment k_t and making no further repayment beyond p_{t-1} (sunk in subperiod $t-1$) guaranteeing it net surplus of at least $u(k_t) - p_{t-1}$. Hence,

$$V_{Bt}^{d} \geqslant u(k_t) - p_{t-1}. \tag{5}$$

Similarly, the seller can "exit" the ex ante stage before making incremental investment Δk_t, guaranteeing it net surplus of at least $p_{t-1} - k_{t-1}$. Hence,

$$V_{St}^{d} \geqslant p_{t-1} - k_{t-1}. \tag{6}$$

Conditions (5) and (6) are bounds on the players' deviation surpluses; actual surpluses depend on off-equilibrium-path strategies, i.e., the punishment for deviation. It is evident that, without loss of generality, the extremal equilibrium involves grim strategy punishments; i.e., in response to a deviation by either player, both cease investment and repayment for the remainder of the ex ante stage. To see this, note the lowest possible values of deviation payoffs V_{Bt}^{d} and V_{St}^{d} subject to conditions (5) and (6) are achieved when (5) and (6) bind, respectively. Playing grim strategies forces (5) and (6) to bind.

We can therefore express the extremal equilibrium as the solution to the problem of choosing sequences K and P to maximize

$$V_{B1} + V_{S1} \tag{7}$$

subject to, for all $t \in \mathbb{N}$,

$$V_{Bt} \geqslant u(k_t) - p_{t-1}, \tag{8}$$

$$V_{St} \geqslant p_{t-1} - k_{t-1}, \tag{9}$$

$$k_t \geqslant k_{t-1}, \tag{10}$$

$$p_t \geqslant p_{t-1}. \tag{11}$$

We will refer to the preceding maximization problem as MAX1. The objective function (7) is total expected social surplus from the perspective of the start of the game. Conditions (8) and (9) are incentive compatibility constraints, ensuring players do not deviate from equilibrium. Implicit in these constraints is that players follow grim strategy punishments for deviation as explained in the previous paragraph. Condition (10) ensures investment is irreversible and condition (11) ensures the seller cannot seize money from the buyer.

MAX1 can be simplified considerably, as Proposition 1 establishes.

Proposition 1. *The extremal equilibrium is given by the sequence K maximizing*

$$\sum_{i=0}^{\infty} \theta(1-\theta)^i \phi(k_{1+i}) \tag{12}$$

subject to, for all $t \in \mathbb{N}$,

$$\sum_{i=0}^{\infty} \theta(1-\theta)^i \phi(k_{t+i}) - \phi(k_t) \geqslant k_t - k_{t-1} \tag{13}$$

and condition (10).

We will refer to the maximization problem in Proposition 1 as MAX2. The proof of this and subsequent results is provided in the appendix; here we will provide a sketch of the proof. The proof has two parts. First, it is a simple matter to show that transfer payments p_t net out of objective function (7) so that $V_{B1} + V_{S1}$ equals the objective function given in (12). Second, it turns out that, without loss of generality, the extremal equilibrium involves exact compensation for the seller's investment expense: $p_t = k_t$ for all $t \in \mathbb{N}$. Exact compensation respects constraint (9), relaxes (8) at least weakly, and has no direct bearing on the objective function since (12) is independent of p_t. Thus, we can eliminate constraint (9) and (11) and substitute $p_t = k_t$ in (8), yielding condition (13).

Following the logic of [7,9], one can see directly from Proposition 1 why investment increments in the extremal equilibrium must gradually decline. The investment sequence must respect the buyer's incentive compatibility constraint (13).

(We will see below that (13) binds in an extremal equilibrium.) The right-hand side of (13)—the subperiod t investment increment—is bounded by the left-hand side of (13)—the expected loss in social surplus from stopping at t rather than proceeding with the equilibrium. Intuitively, the investment increment $k_t - k_{t-1}$ cannot be too large or else the buyer would prefer to avoid repaying at the expense of the loss of future surplus (the buyer fully internalizes social surplus in equilibrium since it exactly repays the seller). As t increases, cumulative investment approaches its limit, the expected loss from stopping decreases, and so the investment increment must correspondingly decrease.

As the next proposition shows, we can derive an explicit solution to MAX2, and thus an explicit solution for the extremal equilibrium.

Proposition 2. *Assume* $\theta \in [0, \hat{\theta})$, *where*

$$\hat{\theta} = \frac{\phi'(0)}{1 + \phi'(0)}. \tag{14}$$

Define $H(\theta)$ *to be the value function*

$$H(\theta) \equiv \max_{k \in \mathbb{R}^+} \left\{ u(k) - \frac{k}{1-\theta} \right\}. \tag{15}$$

There exists an extremal equilibrium with investment sequence $K(\theta) = \{k_t(\theta) \mid t \in \mathbb{N}\}$ *given by*

$$k_t(\theta) \equiv u^{-1}\left(H(\theta) + \frac{k_{t-1}(\theta)}{1-\theta} \right), \tag{16}$$

with exact compensation for the seller's investment $p_t = k_t$ *for all* $t \in \mathbb{N}$.

The proof of Proposition 2 proceeds by solving MAX2 ignoring constraint (10). It is first shown that the incentive compatibility constraints (13) bind at an optimum. Treating (13) as a system of equations, the system can be solved using dynamic programming techniques for the implied investment sequence, yielding $K(\theta)$ as a solution. The proof is completed by showing that the monotonicity constraints (10) are satisfied by $K(\theta)$, so that there was no loss in ignoring the constraints initially.

The next proposition implies that the set of breakdown probabilities $[0, \hat{\theta})$ for which Proposition 2 provides an explicit solution for the extremal equilibrium is nonempty. It further shows that this set of breakdown probabilities covers the unit interval under the Inada condition $u'(0) = \infty$.

Proposition 3. $\hat{\theta} > 0$. *Furthermore*, $\hat{\theta} \to 1$ *as* $u'(0) \to \infty$.

In view of the explicit solution for the extremal equilibrium provided by Proposition 2, we can prove the main results of the paper: first, that for a broad range of cases, the dynamic game with sequential investment and repayment can generate positive investment, and, second, that as the breakdown probability

θ approaches zero, welfare in the dynamic game can come arbitrarily close to the first best.

Proposition 4. *For all* $\theta \in [0, \hat{\theta})$, *there exists an equilibrium with positive investment, i.e., with* $k_t > 0$ *for all* $t \in \mathbb{N}$.

Proposition 5. *There exists an equilibrium such that* $\lim_{\theta \to 0}(V_{B1} + V_{S1}) = \phi(k^*)$; *i.e., the limit of expected social surplus approaches the first best level as the probability of breakdown approaches zero.*

4. Numerical example

Suppose $u(k) = 2\sqrt{k}$ and $\theta = 0.1$ so that the first best investment is $k^* = 1$ and first best welfare equals one. The extremal equilibrium in this numerical example is depicted in Fig. 2. Even though θ is fairly large, and thus there is a fair degree of exogenous friction in the example, the extremal equilibrium comes close to the first best. The figure shows that increments to total investment are initially quite large and become increasingly small as the players progress to later subperiods of investment. This means that there is high probability of recovering a substantial fraction of first best welfare even if the realization of the randomizing device cuts short the relationship between the players. Indeed, even if investment ceased after one subperiod, investment is $k_1 = 0.20$ and welfare is 0.70. That is, 70 percent of the first best welfare is recovered in the second best even in the worst case in which investment ends after only one subperiod. If the relationship survives to at least five periods, cumulative investment is at least $k_5 = 0.49$ and welfare at least 0.91. In the

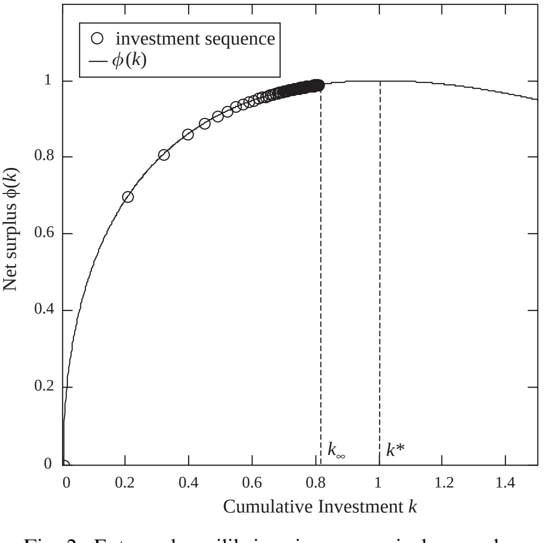

Fig. 2. Extremal equilibrium in a numerical example.

limit as the relationship survives for an increasing number of subperiods, investment approaches $k_\infty = 0.81$ and welfare approaches 0.99. Ex ante expected welfare in the extremal equilibrium is 0.90; i.e., on average, 90 percent of the first best welfare is recovered in the extremal equilibrium.

5. Conclusion

We studied a situation in which the hold-up problem is quite severe in that a seller provides a buyer with an investment benefitting the buyer, the buyer has control rights over the benefit of investment, and contracts over investment and trade are fully incomplete. We proposed a noncontractual solution to the hold-up problem. The solution is a game involving investment by installments by the supplier and reimbursement by the buyer after each installment. The supplier's threat of withholding future investment induces the buyer to make the necessary repayments. In the limit as frictions disappear (in the form of a shrinking probability θ that the relationship between the players breaks down), social welfare in the extremal equilibrium approaches the first best.

Broadly speaking, our results might be thought of as solving the hold-up problem by constructing a series of spot markets for investment: investments are made and reimbursed without recourse to contracts. One has to be precise about what is meant by a spot market to avoid trivializing the problem, as would be the case if it were assumed that spot markets allowed for the simultaneous exchange of investment for money. We do not assume simultaneous exchange here: in our model, payments can only be made after investment is sunk. Yet we still show that dividing investment into installments is beneficial.

The crucial assumption for our results is that it is costless for the buyer to observe and repay any incremental investment by the seller. A known, fixed cost of observing investment, restarting investment, or other such friction would cause the equilibrium to unravel: in late enough subperiods the future gains from cooperation would eventually be swamped by the finite cost, implying that only a finite number of periods of cooperation would be possible. On the other hand, if players are slightly altruistic in that they are willing to make payments or investments smaller than a certain amount without compensation (e.g., the seller is willing to hammer in the last nail for free), then even a finite game may not unravel. There is some empirical support for this last point: games with long but finite horizons have been shown in experimental settings not to unravel completely. For example, in experiments on the centipede game (e.g., [21]), a game in which subgame perfect equilibrium is for the game to end immediately with the first player taking the money, experimental subjects turn out to proceed for a number of periods, letting the pie grow along the way.

Our results are robust to several modifications of the model, which space constraints prevent us from discussing in more detail. (a) We assumed one particular control rights regime: a buyer control regime in which the buyer could not be excluded from enjoying the benefits of ex ante investment. This assumption was pedagogically convenient since it produced the stark outcome of zero investment in the benchmark, static model. Our results continue to hold if one instead assumed a seller control regime in which the seller's permission were required for the buyer to enjoy investment benefits. There would still be a hold-up problem which would be ameliorated by gradual investment. (b) We assumed the friction took the form of a constant per-period probability of a breakdown in the seller–buyer relationship. Our results continue to hold if the probability is allowed to vary with the length of the subperiod or alternatively with the amount of investment undertaken in the subperiod. (c) Our analysis would be similar if one replaced the probability of breakdown with a different sort of friction—discounting—in a modified model along the lines of [6] in which the buyer enjoys the benefit of cumulative investment along the path while investment is undertaken.

The main goal of our work is to introduce the idea that investment by installment may be a useful way to address the hold-up problem in certain applications, and to outline the conditions under which such a solution might be plausible. We do not suggest that we have eliminated the hold-up problem generally. Investment may be lumpy, monitoring by the buyer may involve technological and opportunity costs, a host of factors may prevent parties from efficiently subdividing investment into installments.

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Appendix A

Proof of Proposition 1. To show MAX1 and MAX2 are equivalent, we first show their objective functions are the same. The objective function from MAX1, (7), is

$$\begin{aligned} &V_{B1} + V_{S1} \\ &\quad = \sum_{i=0}^{\infty} \theta(1-\theta)^i [u(k_{1+i}) - p_{1+i}] + \sum_{i=0}^{\infty} \theta(1-\theta)^i [p_{1+i} - k_{1+i}] \\ &\quad = \sum_{i=0}^{\infty} \theta(1-\theta)^i \phi(k_{1+i}), \end{aligned} \tag{A.1}$$

the objective function in MAX2, (12).

Second, we show that the solution to MAX1 involves $p_t = k_t$ for all $t \in \mathbb{N}$ without loss of generality. Ignore constraint (11) for the moment; we will verify at the end that it is satisfied by the resulting solution. Given the objective function can be rewritten as in (12), and the repayments P do not appear in this rewritten objective function, P only appears in incentive compatibility constraints (8) and (9). But (8) can be written equivalently as

$$V_{Bt} + (V_{St} - V_{St}) \geqslant u(k_t) - p_{t-1} + (k_{t-1} - k_{t-1}),$$

which is equivalent to

$$\sum_{i=0}^{\infty} \theta(1-\theta)^i \phi(k_{t+i}) - (V_{St} - p_{t-1} + k_{t-1}) \geqslant u(k_t) - k_{t-1} \tag{A.2}$$

since

$$V_{Bt} + V_{St} = \sum_{i=0}^{\infty} \theta(1-\theta)^i \phi(k_{t+i}). \tag{A.3}$$

Note (A.3) can be derived by calculations similar to those in (A.1). Now, (A.2) is relaxed by reducing the expression in parentheses $V_{St} - p_{t-1} + k_{t-1}$. But rewriting (9) shows $V_{St} - p_{t-1} + k_{t-1} \geqslant 0$. Setting $p_t = k_t$ for all $t \in \mathbb{N}$ minimizes $V_{St} - p_{t-1} + k_{t-1}$ subject to the nonnegativity constraint. To see this, note $p_t = k_t$ for all $t \in \mathbb{N}$ implies $V_{St} = 0$ for all $t \in \mathbb{N}$, implying $V_{St} - p_{t-1} + k_{t-1} = 0$ for all $t \in \mathbb{N}$. In sum, we can substitute $p_t = k_t$ for all $t \in \mathbb{N}$ in MAX1 and ignore (9), as MAX2 reflects.

Finally, we need to verify that the omitted constraint (11) is satisfied by the solution to MAX2. But (10) together with $p_t = k_t$ for all $t \in \mathbb{N}$ ensures (11) holds. □

Proof of Proposition 2. By Proposition 1, the solution to MAX2 implements the extremal equilibrium. The proof proceeds by finding an explicit solution to MAX2. Ignore constraint (10) for the moment. We will verify below that (10) is satisfied for all $t \in \mathbb{N}$ by the resulting solution. Fix $\{k_i \mid i \in \mathbb{N}, i \neq t\}$. We claim that the k_t maximizing (12) forces constraint (13) to hold with equality. To see this, rewrite (13) after some algebraic manipulations as follows:

$$k_{t-1} + \sum_{i=1}^{\infty} \theta(1-\theta)^i \phi(k_{t+i}) \geqslant (1-\theta)u(k_t) + \theta k_t. \tag{IC$_t$}$$

The right-hand side of (IC$_t$) is increasing in k_t, and k_t does not appear on the left-hand side. The highest value of k_t subject to (IC$_t$) can thus be found by treating (IC$_t$) as an equality. Assume $k_t < k^*$ for all $t \in \mathbb{N}$. (Again, we will verify below that this additional constraint is satisfied by the resulting solution.) Then higher values of k_t are desirable since the objective function (12) is increasing in k_t.

A remaining complication is that k_t appears not just in constraint (IC$_t$) but also in constraints (IC$_1$), (IC$_2$), …, (IC$_{t-1}$). Now ϕ is strictly increasing on $[0, k^*)$. To see this, note $u''(k) < 0$ by assumption, so $\phi''(k) = u''(k) < 0$ for all $k \in \mathbb{R}^+$. Thus, $\phi'(k) > \phi'(k^*)$ for all $k \in [0, k^*)$. But $\phi'(k^*) = 0$ since k^* is assumed to be an interior maximizer. Therefore, $\phi'(k) > 0$ for all $k \in [0, k^*)$. This fact, combined with the fact that $k_t < k^*$ by assumption, implies that constraints $(\mathrm{IC}_1), (\mathrm{IC}_2), \ldots, (\mathrm{IC}_{t-1})$ are relaxed if k_t is increased.

Treating (IC_t) as a system of equations, the system can be solved using dynamic programming techniques for the implied investment sequence. We will show the solution equals $K(\theta)$. In particular, substituting $t-1$ into Eq. (1), we have

$$V_{\mathrm{B}t-1} = \theta\phi(k_{t-1}) + (1-\theta)V_{\mathrm{B}t}. \tag{A.4}$$

Treating incentive compatibility constraint (13) for subperiod t as an equality, and rearranging terms, yields

$$V_{\mathrm{B}t} = u(k_t) - k_{t-1}. \tag{A.5}$$

Similarly, treating the incentive compatibility constraint (13) for subperiod $t-1$ as an equality, and rearranging terms, yields

$$V_{\mathrm{B}t-1} = u(k_{t-1}) - k_{t-2}. \tag{A.6}$$

Substituting for $V_{\mathrm{B}t}$ from Eq. (A.5) and for $V_{\mathrm{B}t-1}$ from (A.6) into Eq. (A.4), we have, after suitably rearranging terms, for all $t \in \mathbb{N}$,

$$u(k_t) = u(k_{t-1}) + \frac{k_{t-1} - k_{t-2}}{1-\theta}. \tag{A.7}$$

We argue by induction that Eq. (A.7) implies, for all $t \in \mathbb{N}$,

$$u(k_t) = u(k_1) + \frac{k_{t-1}}{1-\theta}. \tag{A.8}$$

It is evident that (A.8) holds for $t = 1$ since $k_0 = 0$, so the formula simply gives $u(k_1) = u(k_1)$. Assume as the inductive hypothesis that (A.8) holds for all t. We will show that (A.8) also holds for $t+1$. We have

$$\begin{aligned} u(k_{t+1}) &= u(k_t) + \frac{k_t - k_{t-1}}{1-\theta} \\ &= u(k_1) + \frac{k_t}{1-\theta}. \end{aligned}$$

The first line holds by (A.7). The second line holds by applying the inductive hypothesis.

If it could be shown $k_1 = H(\theta)$, then Eq. (A.8) gives the same recursive formula as Eq. (16). It remains to show that $k_1 = H(\theta)$ in an extremal equilibrium. Pinning down the initial value k_1 is equivalent to pinning down the terminal value $k_\infty = \lim_{t\to\infty} k_t$. These values can be pinned down by returning to the objective of maximizing (12). Given incentive compatibility binds for subperiod $t = 1$,

$$\sum_{i=0}^{\infty} \theta(1-\theta)^i \phi(k_{1+i}) = u(k_1) - k_0 = u(k_1). \tag{A.9}$$

Maximizing (12) subject to (A.8) for all $t \in \mathbb{N}$ is thus equivalent to maximizing $u(k_1)$ subject to, for all $t \in \mathbb{N}$,

$$u(k_1) = u(k_t) - \frac{k_{t-1}}{1-\theta}. \tag{A.10}$$

The terminal ($t \to \infty$) condition implied by Eq. (A.10) is

$$u(k_1) = u(k_\infty) - \frac{k_\infty}{1-\theta}.$$

Given k_∞ is a free parameter, we can choose it to maximize $u(k_1)$:

$$u(k_1) = \max_{k \in \mathbb{R}^+} \left\{ u(k) - \frac{k}{1-\theta} \right\} = H(\theta).$$

This yields the value of $k_1(\theta)$ in Eq. (16).

The proof is completed by showing that the constraints (10) are satisfied by $K(\theta)$, so that there was no loss in ignoring the constraints initially. In fact, though (10) require only that $K(\theta)$ be monotone nondecreasing, we will prove a stronger result, namely $K(\theta)$ is monotonically increasing, by induction on t. First, we will show $k_1(\theta) < k_2(\theta)$:

$$\begin{aligned} k_2(\theta) &= u^{-1}\left(u(k_1(\theta)) + \frac{k_1(\theta)}{1-\theta}\right) \\ &> u^{-1}(u(k_1(\theta))) \\ &= k_1(\theta). \end{aligned}$$

The first line holds by definition of $k_2(\theta)$ from Eq. (16). The second line holds because u^{-1} is an increasing function and because a positive term has been eliminated. Assume as the inductive hypothesis that $k_{t-1}(\theta) < k_t(\theta)$. We will show $k_t(\theta) < k_{t+1}(\theta)$:

$$\begin{aligned} k_{t+1}(\theta) &= u^{-1}\left(u(k_1(\theta)) + \frac{k_t(\theta)}{1-\theta}\right) \\ &> u^{-1}\left(u(k_1(\theta)) + \frac{k_{t-1}(\theta)}{1-\theta}\right) \\ &= k_t(\theta). \end{aligned}$$

The first line holds by definition of $k_{t+1}(\theta)$ from Eq. (16). The second line holds since u^{-1} is increasing and $k_{t-1}(\theta) < k_t(\theta)$. The last line holds by definition of $k_t(\theta)$ from Eq. (16). This establishes $k_t(\theta) < k_{t+1}(\theta)$ for all $t \in \mathbb{N}$. □

Proof of Proposition 3. First, we prove $\hat{\theta} > 0$. Now u is strictly concave by assumption, so ϕ is also strictly concave. Therefore $0 = \phi'(k^*) < \phi'(0)$, where the equality holds since k^* is an interior maximizer and the inequality holds since ϕ is concave and $k^* > 0$. But by definition of $\hat{\theta}$ from (14), $\phi'(0) > 0$ implies $\hat{\theta} > 0$.

Next, we prove $\hat{\theta} \to 1$ as $u'(0) \to \infty$. Now $\phi'(0) = u'(0) - 1$. Therefore, by (14), $\hat{\theta} = [u'(0) - 1]/u'(0) \to 1$ as $u'(0) \to \infty$. □

Proof of Proposition 4. Suppose $\theta \in (0, \hat{\theta})$. Then

$$\begin{aligned} k_1(\theta) &= u^{-1}\left(H(\theta) + \frac{k_0(\theta)}{1-\theta}\right) \\ &= u^{-1}\left(\max_{k \in \mathbb{R}^+}\left\{u(k) - \frac{k}{1-\theta}\right\}\right) \\ &> u^{-1}\left(\max_{k \in \mathbb{R}^+}\left\{u(k) - \frac{k}{1-\hat{\theta}}\right\}\right) \\ &\geqslant u^{-1}(u(0)) \\ &= 0. \end{aligned}$$

The first line holds by definition from Eq. (16). The second line holds by the definition $k_0(\theta) = 0$ and the definition of $H(\theta)$ from Eq. (15). The third line holds since the right-hand side is decreasing in θ and $\theta < \hat{\theta}$ by assumption. The fourth line holds since zero is in the set of maximizers. We have therefore demonstrated the existence of an equilibrium with $k_1 > 0$. In the proof of Proposition 2, we showed that $K(\theta)$ is strictly monotonic. Hence, $k_t > 0$ for all $t \in \mathbb{N}$. □

Proof of Proposition 5. In the extremal equilibrium of Proposition 2,

$$\begin{aligned} V_{B1} + V_{S1} &= V_{B1} \\ &= u(k_1(\theta)) - k_0(\theta) \\ &= u\left(u^{-1}\left(H(\theta) + \frac{k_0(\theta)}{1-\theta}\right)\right) \\ &= H(\theta). \end{aligned}$$

The first line holds since the extremal equilibrium involves exact repayment, implying $p_t = k_t$ for all $t \in \mathbb{N}$, in turn implying $V_{St} = 0$ for all $t \in \mathbb{N}$. The second line holds since the incentive compatibility constraint (13) was shown to bind in the extremal equilibrium. The third line holds by the definition $k_0(\theta) = 0$ and the definition of $k_1(\theta)$ from Eq. (16). The last line holds again by the definition $k_0(\theta) = 0$.

Therefore, we have

$$\begin{aligned} \lim_{\theta \to 0} (V_{B1} + V_{S1}) &= \lim_{\theta \to 0} H(\theta) \\ &= \max_{k \in \mathbb{R}^+} \{u(k) - k\} \\ &= \phi(k^*). \end{aligned}$$

The first line follows from the derivations in the preceding paragraph. The second line follows from taking the limit as $\theta \to 0$ in the definition of $H(\theta)$ in Eq. (15). The last line follows from the definition of k^*. □

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