Online Appendixes
Optimal Vaccine Subsidies for Epidemic Diseases

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This document contains a series of online appendixes supplementing the published article. The appendixes provide analytical proofs or extensions omitted for space considerations. Appendix A1 provides proofs, omitted from the text, of lemmas and propositions. The proofs are streamlined by the inclusion of additional lemmas, stated and proved in Appendix A1. Appendix A2 provides additional documentation for the assumption made in the calibration section for taking the limit $\tilde{c} \downarrow 0$. Appendix A3 analyzes Cournot competition among n firms. This analysis nests perfect competition studied in the article in the limit $n \uparrow \infty$ and also nests monopoly studied in the article setting $n = 1$. Appendix A4 extends the analysis of homogeneous consumers to allow consumers to vary in disease harm H_i. Appendix A5 extends the model to allow for a second preventive technology, competitively supplied, possibly interpreted as social distancing. We show that the basic comparative-static results for vaccine-market equilibrium are essentially unchanged.

Appendix A1. Proofs

Proof of Lemma 1

We begin by proving the claims about $I_t(Q)$. Substituting (10) into (3) and rearranging yields

$$\frac{\dot{I}_t(Q)}{I_t(Q)} = \alpha[\mathcal{R}_0 S_t(Q) - 1]. \tag{A1}$$

Recognizing the left-hand side as $\partial \ln I_t(Q)/\partial t$ and integrating yields

$$\int_0^t \frac{\partial \ln I_\tau(Q)}{\partial \tau} d\tau = \int_0^t \alpha[\mathcal{R}_0 S_\tau(Q) - 1] d\tau. \tag{A2}$$

Invoking the Fundamental Theorem of Calculus, taking exponentials, and rearranging yields

$$I_t(Q) = I_0(Q) \exp\left(\int_0^t \alpha[\mathcal{R}_0 S_\tau(Q) - 1] d\tau\right). \tag{A3}$$

Since $I_0(Q) = \hat{I}_0 > 0$ by assumption, $I_t(Q)$ is the product of two positive factors.

Turn next to proving the claims about $S_t(Q)$. Rearranging (4), $I_t(Q) = \dot{R}_t(Q)/\alpha$. Substituting into (2) and rearranging yields $\dot{S}_t(Q)/S_t(Q) = -(\beta/\alpha)\dot{R}_t(Q) = -\mathcal{R}_0 \dot{R}_t(Q)$ by (10). Recognizing $\dot{S}_t(Q)/S_t(Q) = \partial \ln S_t(Q)/\partial t$ and integrating between $t' \geq 0$ and $t'' \geq t'$ yields

$$\int_{t'}^{t''} \frac{\partial \ln S_\tau(Q)}{\partial \tau} d\tau = -\int_{t'}^{t''} \frac{1}{\alpha} \dot{R}_\tau(Q) d\tau. \tag{A4}$$

Invoking the Fundamental Theorem of Calculus, taking exponentials, and rearranging yields

$$S_{t''}(Q) = S_{t'}(Q)e^{\mathcal{R}_0[R_{t'}(Q)-R_{t''}(Q)]}. \tag{A5}$$

Substituting $t' = 0$ and $t'' = t$ into (A5) yields

$$S_t(Q) = S_0(Q)e^{\mathcal{R}_0[R_0(Q)-R_t(Q)]}. \tag{A6}$$

Now $S_0(Q) = \hat{S}_0 - \theta Q > \hat{S}_0 - Q \geq 0$, where the first step holds by (6), the second by $\theta < 1$, and the third by $Q \in [0, \hat{S}_0]$. The right-hand side of (A6) is thus the product of two positive factors. *Q.E.D.*

Proof of Lemma 2

Substituting $I_t(Q) > 0$ into (4) yields $\dot{R}_t(Q) > 0$, implying $R_{t''}(Q) > R_{t'}(Q)$ for $t'' > t'$, implying $e^{\mathcal{R}_0[R_{t'}(Q)-R_{t''}(Q)]} < 1$. Since $S_{t'}(Q) > 0$ by Lemma 1, $S_{t''}(Q) \leq S_{t'}(Q)$ by (A5). *Q.E.D.*

Proof of Lemma 3

Since $I_t(Q) > 0$ by Lemma 1, the sign of $\dot{I}_t(Q)$ is determined by the value of $\mathcal{R}_0 S_t(Q)$ relative to 1 by (A1). First, suppose $\mathcal{R}_0 S_0(Q) \leq 1$. Consider any $t > 0$. Lemma 2 implies $S_t(Q) < S_0(Q)$, in turn implying $\mathcal{R}_0 S_t(Q) < \mathcal{R}_0 S_0(Q) \leq 1$. Substituting $\mathcal{R}_0 S_t(Q) < 1$ into (A1) implies $\dot{I}_t(Q) < 0$.

Next, suppose $\mathcal{R}_0 S_0(Q) > 1$. Substititing $t = 0$ into (A1) implies $\dot{I}_0(Q) > 0$. By Martcheva (2015, p. 13), $I_\infty(Q) = 0$. Since $I_0(Q) > 0$ by Lemma 1, $\dot{I}_t(Q) < 0$ for some $t > 0$. By continuinity, $\dot{I}_T(Q) = 0$ for some $T > 0$. Setting (A1) equal to 0 yields $\mathcal{R}_0 S_T(Q) = 1$. Since $S_t(Q)$ is strictly decreasing, $\mathcal{R}_0 S_t(Q) > \mathcal{R}_0 S_T(Q) = 1$ for all $t \in [0, T)$, implying $\dot{I}_t(Q) > 0$ for all $t \in [0, T)$ by (A1). One can similarly show $\dot{I}_t(Q) < 0$ for all $t > T$. Thus, $I_T(Q)$ is the maximum infection rate. *Q.E.D.*

Proof of Lemma 4

See Martcheva (2015, p. 13) for a proof that $I_\infty(Q) = 0$. Martcheva (2015, p. 12) argues that the fact that $S_t(Q)$ is positive and montone implies that the limit $S_\infty(Q)$ exists.

To prove the remaining claim in the lemma, take the limit $t \uparrow \infty$ in (A6):

$$S_\infty(Q) = S_0(Q)e^{\mathcal{R}_0[R_0(Q)-R_\infty(Q)]}. \tag{A7}$$

By Lemma 1, $S_0(Q) > 0$. The proof of Lemma 2 showed that $R_t(Q)$ is strictly increasing in t. Thus $R_\infty(Q) > R_0(Q)$, implying $S_\infty(Q) < S_0(Q)$ by (A7). *Q.E.D.*

Proof of Lemma 5

First, suppose $\mathcal{R}_0 S_0(Q) \leq 1$. Then $\mathcal{R}_0 S_\infty(Q) < 1$ because $S_t(Q)$ is strictly decreasing in t by Lemma 2. Next, suppose $\mathcal{R}_0 S_0(Q) > 1$. The proof of Lemma 3 established the existence of $T > 0$ such that $\mathcal{R}_0 S_T(Q) = 1$. Since $S_t(Q)$ is strictly decreasing by Lemma 2, we have $\mathcal{R}_0 S_\infty(Q) < \mathcal{R}_0 S_T(Q) = 1$. *Q.E.D.*

Proof of Lemma 6

Substituting (2) and (10) into (3) yields

$$\dot{I}_t(Q) = \frac{\dot{S}_t(Q)}{\mathcal{R}_0 S_t(Q)} - \dot{S}_t(Q). \tag{A8}$$

Integrating (A8) over $t \in [0, \infty)$ and applying the Fundamental Theorem of Calculus,

$$I_\infty(Q) - I_0(Q) = \frac{1}{\mathcal{R}_0}[\ln S_\infty(Q) - \ln S_0(Q)] - S_\infty(Q) + S_0(Q). \tag{A9}$$

Substituting $I_0(Q) = \hat{I}_0$ by (7), noting $I_\infty(Q) = 0$ by Lemma 4, and rearranging yields

$$\ln S_\infty(Q) - \mathcal{R}_0 S_\infty(Q) = \ln S_0(Q) - \mathcal{R}_0[\hat{I}_0 + S_0(Q)]. \tag{A10}$$

Further substituting $S_0(Q) = \hat{S}_0 - Q$ from (6) yields (11).

To derive (12), exponentiating both sides of (A10) and rearranging yields

$$S_\infty(Q) = \left\{ S_0(Q) e^{-\mathcal{R}_0[\hat{I}_0 + S_0(Q)]} \right\} e^{\mathcal{R}_0 S_\infty(Q)}, \tag{A11}$$

or, equivalently,

$$x = be^{ax}, \tag{A12}$$

where $x = S_\infty(Q)$, $a = \mathcal{R}_0$, and $b = S_0(Q)e^{-\mathcal{R}_0[\hat{I}_0 + S_0(Q)]}$. It is well-known that (A12) has solution $x = -\bar{L}(-ab)/a = |\bar{L}(-ab)|/a$, where the second equality holds if $a, b > 0$ implying $\bar{L}(-ab) < 0$. Substituting for x, a, and b in this solution as well as $S_0(Q) = \hat{S}_0 - Q$ from (6) yields (12).

Equation (A12) also has a solution in terms of the lower branch of the Lambert W function, $x = -\underline{L}(-ab)/a$. We reject this solution because it exceeds 1, out of bounds for $S_\infty(Q)$. *Q.E.D.*

Additional Lemmas

We state and prove two additional lemmas, which draw on previous results, which will help streamline the subsequent proofs.

Lemma 7. $\lim_{\mathcal{R}_0 \downarrow 0} S_\infty(Q) = \hat{S}_0 - \theta Q$ *and* $\lim_{\mathcal{R}_0 \downarrow 0} [\mathcal{R}_0 S_\infty(Q)] = 0$.

Proof. The first limit can be shown to hold by substituting $\mathcal{R}_0 = 0$ into (11). The second limit then follows: $\lim_{\mathcal{R}_0 \downarrow 0} [\mathcal{R}_0 S_\infty(Q)] = (\hat{S}_0 - \theta Q) \lim_{\mathcal{R}_0 \downarrow 0} \mathcal{R}_0 = 0$. *Q.E.D.*

Lemma 8. $\lim_{\mathcal{R}_0 \uparrow \infty} S_\infty(Q) = \lim_{\mathcal{R}_0 \uparrow \infty} [\mathcal{R}_0 S_\infty(Q)] = 0$.

Proof. We will verify the second limit; the first limit is an immediate consequence. We have

$$\lim_{\mathcal{R}_0 \uparrow \infty} [\mathcal{R}_0 S_\infty(Q)] = \lim_{\mathcal{R}_0 \uparrow \infty} \left| \bar{L}\left(-\mathcal{R}_0 S_0(Q) e^{-\mathcal{R}_0[\hat{I}_0 + S_0(Q)]} \right) \right| \tag{A13}$$

$$= \left| \bar{L}\left(-S_0(Q) \lim_{\mathcal{R}_0 \uparrow \infty} \frac{\mathcal{R}_0}{e^{\mathcal{R}_0(\hat{I}_0 + S_0(Q))}} \right) \right| \tag{A14}$$

$$= |\bar{L}(0)|. \tag{A15}$$

Equation (A13) follows by taking limits in (12), (A14) is a simple rearrangement, and (A15) follows from application of l'Hôpital's Rule. As is well known for the Lambert W function, $\bar{L}(0) = 0$. *Q.E.D.*

Lemma 9. $\partial S_\infty(Q)/\partial \mathcal{R}_0 < 0$.

Proof. The Implicit Function Theorem can be applied to (11) to compute the derivative

$$\frac{\partial S_\infty(Q)}{\partial \mathcal{R}_0} = \frac{-S_\infty(Q)}{1-\mathcal{R}_0 S_\infty(Q)}[\hat{I}_0 + S_0(Q) - S_\infty(Q)]. \tag{A16}$$

The first factor is negative by Lemma 5. The factor in square brackets is positive since $\hat{I}_0 + S_0(Q) - S_\infty(Q) > S_0(Q) - S_\infty(Q) > 0$, where the first inequality follows from $\hat{I}_0 > 0$ and the second by Lemma 4. *Q.E.D.*

Lemma 10. $\partial \Phi(Q)/\partial Q < 0$.

Proof. Differentiating (14) and substituting from (13) yields

$$\frac{\partial \Phi(Q)}{\partial Q} = \frac{-\theta \Phi(Q) \mathcal{R}_0 S_\infty(Q)}{S_0(Q)[1-\mathcal{R}_0 S_\infty(Q)]}, \tag{A17}$$

which is negative by Lemma 5. *Q.E.D.*

Verification of Table 1 Entries

The equilibrium condition is $P_c^* = c$. Firms earn no profit under perfect competition: $\Pi_c^* = 0$. No consumers purchase in case (i), implying $Q_c^* = 0$. All susceptibles purchase in case (iii), implying $Q_c^* = \hat{S}_0$. In case (ii), Q_c^* can be found by substituting $P_c^* = c$ in equation (17).

To find $R_\infty(Q_c^*)$, note $R_\infty(Q_c^*) = 1 - I_\infty(Q_c^*) - S_\infty(Q_c^*) - \theta Q_c^* = 1 - S_\infty(Q_c^*) - \theta Q_c^*$ since $I_\infty(Q_c^*) = 0$. Substituting $Q_c^* = 0$ gives the entry for $R_\infty(Q_c^*)$ in case (i), and substituting $Q_c^* = \hat{S}_0$ gives the entry for $R_\infty(Q_c^*)$ in case (iii). To find $R_\infty(Q_c^*)$ in case (ii), set $c = MPB(Q_c^*)$ in equation (15) and rearrange, yielding

$$S_\infty(Q_c^*) = (1-\tilde{c})(\hat{S}_0 - \theta Q_c^*) \tag{A18}$$

Substituting (A18) into $R_\infty(Q_c^*) = 1 - S_\infty(Q_c^*) - \theta Q_c^*$ and rearranging yields $R_\infty(Q_c^*) = 1 - (1-\tilde{c})\hat{S}_0 - \tilde{c}\theta Q_c^*$. Substituting from the table entry for Q_c^* yields the table entry for $R_\infty(Q_c^*)$.

Substituting $Q_c^* = 0$ in (15) gives MPB_c^* in case (i), and substituting $Q_c^* = \hat{S}_0$ in (15) gives MPB_c^* in case (iii). For some but not all consumers to purchase in case (ii) requires $MPB_c^* = c$.

Substituting $Q_c^* = 0$ in (22) gives MSB_c^* in case (i), and substituting $Q_c^* = \hat{S}_0$ in (22) gives MSB_c^* in case (iii). Substituting from (A18) into (22) yields MSB_c^* in case (ii).

The table entries for MEX_c^* can be obtained by subtracting other table entries: $MEX_c^* = MSB_c^* - MPB_c^*$. To derive the table table entries for W_c^*, by definition $W_c^* = SB_c^* - cQ_c^* = H[1 - R_\infty(Q_c^*)] - cQ_c^*$, where the second equation follows from (20). Substituting other table entries into this equation gives the table entries for W_c^*. *Q.E.D.*

Proof of Proposition 1

Results for P_c^*, Π_c^*, and W_c^*. The results for P_c^* and Π_c^* are obvious from Table 1. The comparative statics for W_c^* are also obvious from inspection of the table in view of (A16).

Results for Q_c^*. To show Q_c^* is weakly increasing, it can be verified that it is continuous at thresholds $\mathcal{R}_0'$ and $\mathcal{R}_0''$ defined in (24)–(25). In case (ii), $\partial Q_c^*/\partial \mathcal{R}_0 = -\ln(1-\tilde{c})/\theta\tilde{c}\mathcal{R}_0^2 > 0$. Hence, Q_c^* is weakly increasing in $\mathcal{R}_0$ for all $\mathcal{R}_0 > 0$ and strictly increasing for $\mathcal{R}_0$ in the interior of case (ii).

Results for MPB_c^*. To show MPB_c^* is weakly increasing, start with case (i). Differentiating the table entry,

$$\frac{\partial MPB_c^*}{\partial \mathcal{R}_0} = -\left(\frac{\theta H}{\hat{S}_0}\right)\frac{\partial S_\infty(0)}{\partial \mathcal{R}_0}. \tag{A19}$$

By Lemma 9, $\partial S_\infty(Q)/\partial \mathcal{R}_0 < 0$ for all $Q \in [0, \hat{S}_0]$, including $Q_c^* = 0$, implying (A19) is positive. In case (ii), MPB_c^* is constant. Differentiating the table entry in cases (iii) and (iv),

$$\frac{\partial MPB_c^*}{\partial \mathcal{R}_0} = -\left[\frac{\theta H}{(1-\theta)\hat{S}_0}\right]\frac{\partial S_\infty(\hat{S}_0)}{\partial \mathcal{R}_0}, \tag{A20}$$

which is negative since $\partial S_\infty(Q)/\partial \mathcal{R}_0 < 0$ by Lemma 9 for all $Q \in [0, \hat{S}_0]$, including $Q_c^* = \hat{S}_0$.

The last step in deriving comparative statics for MPB_c^* is to show it is continuous at both endpoints of case (ii). Now $MPB(Q)$ is continuous in Q because it is differentiable in Q by (16). Further, $MPB(Q)$ is continuous in $\mathcal{R}_0$ because $S_\infty(Q)$ is differentiable in $\mathcal{R}_0$ by (A16). Since Q_c^* is continuous at both endpoints of case (ii) as argued in the first paragraph of this proof, we have that MPB_c^* is continous at $\mathcal{R}_0'$ and $\mathcal{R}_0''$.

Results for $R_\infty(Q_c^*)$. To derive the comparative statics for $R_\infty(Q_c^*)$, combining the table entries with Lemma 9 shows $R_\infty(Q_c^*)$ is increasing in $\mathcal{R}_0$ in case (i) as well as cases (iii) and (iv). The table entry is obviously decreasing in $\mathcal{R}_0$ in case (ii). We thus have that $R_\infty(Q_c^*)$ attains a local maximum at $\mathcal{R}_0'$ if we can establish that $R_\infty(Q_c^*)$ is continuous at $\mathcal{R}_0'$. Using the table entry for $R_\infty(Q_c^*)$ in case (i),

$$\lim_{\mathcal{R}_0 \uparrow \mathcal{R}_0'} R_\infty(Q_c^*) = 1 - \lim_{\mathcal{R}_0 \uparrow \mathcal{R}_0'} S_\infty(0) = 1 - \left(1 - \lim_{\mathcal{R}_0 \uparrow \mathcal{R}_0'} \frac{MPB_c^*}{\theta H}\right)\hat{S}_0 = 1 - (1-\tilde{c})\hat{S}_0. \tag{A21}$$

The second equality follows from the table entry for MPB_c^* in case (i): $MPB_c^* = \theta H \Phi(0) = \theta H[1 - S_\infty(0)/\hat{S}_0]$ by (14). The third equality follows from the continuity of MPB_c^* at $\mathcal{R}_0'$, allowing us to substitute the table entry for $MPB_c^* = c$ in case (ii). Using the table entry for $R_\infty(Q_c^*)$ in case (ii),

$$\lim_{\mathcal{R}_0 \downarrow \mathcal{R}_0'} R_\infty(Q_c^*) = 1 - \hat{S}_0 - \hat{I}_0 + \frac{1}{\mathcal{R}_0'}|\ln(1-\tilde{c})| = 1 - (1-\tilde{c})\hat{S}_0. \tag{A22}$$

The equality between (A21) and (A22) proves the continuity of $R_\infty(Q_c^*)$ at $\mathcal{R}_0'$.

Since $R_\infty(Q_c^*)$ is increasing in $\mathcal{R}_0$ in cases (iii) and (iv), the other candidate for a supremum is

$$\lim_{\mathcal{R}_0 \uparrow \infty} R_\infty(Q_c^*) = 1 - \theta\hat{S}_0. \tag{A23}$$

This equality follows from taking the limit $\mathcal{R}_0 \uparrow \infty$ of the table entry in cases (iii) and (iv) and noting that $\lim_{\mathcal{R}_0 \uparrow \infty} S_\infty(\hat{S}_0) = 0$ by Lemma 8. The local maximum is thus a global maximum if and only if $1 - (1-\tilde{c})\hat{S}_0 \geq 1 - \theta\hat{S}_0$. Rearranging gives $\tilde{c} \geq 1 - \theta$.

Results for MSB_c^*. To provide a roadmap for the analysis, we first look at cases (i) and (ii) and show that MSB_c^* has a unique local maximum over $\mathcal{R}_0 \le \mathcal{R}_0''$. Furthermore, this restricted local maximum is the restricted global maximum over $\mathcal{R}_0 \le \mathcal{R}_0''$. We then look at cases (iii) and (iv) and show that MSB_c^* has at most one restricted local maximum over $\mathcal{R}_0 > \mathcal{R}_0''$. If no restricted local maximum exists there, then we show that the restricted maximum over $\mathcal{R}_0 \le \mathcal{R}_0''$ is the global maximum over all $\mathcal{R}_0 > 0$. If a restricted local maximum exists in cases (iii) and (iv), then either it or the restricted local maximum over $\mathcal{R}_0 \le \mathcal{R}_0''$ is the global maximum. This establishes, in sum, that MSB_c^* has at most two local maxima, one of which is the global maximum.

To prove that MSB_c^* has a unique local restricted maximum over $\mathcal{R}_0 \le \mathcal{R}_0''$, we will show that MSB_c^* is increasing in a neighborhood around 0, quasiconcave for all $\mathcal{R}_0 \in (0, \mathcal{R}_0']$, continuous at $\mathcal{R}_0'$, and decreasing for all $\mathcal{R}_0 \in (\mathcal{R}_0', \mathcal{R}_0'')$. The arguments are made in reverse order. It is clear from inspection of Table 1 that MSB_c^* is decreasing for $\mathcal{R}_0 \in (\mathcal{R}_0', \mathcal{R}_0'')$. To show MSB_c^* is continuous at $\mathcal{R}_0'$, using the table entry for MSB_c^* in case (i),

$$\lim_{\mathcal{R}_0 \uparrow \mathcal{R}_0'} MSB_c^* = \frac{\lim_{\mathcal{R}_0 \uparrow \mathcal{R}_0'} MPB_c^*}{1 - \mathcal{R}_0'(1 - \lim_{\mathcal{R}_0 \uparrow \mathcal{R}_0'} MPB_c^*/\theta H)\hat{S}_0} = \frac{\theta H \tilde{c}(\hat{I}_0 + \tilde{c}\hat{S}_0)}{\hat{I}_0 + \tilde{c}\hat{S}_0 + (1-\tilde{c})\hat{S}_0 \ln(1-\tilde{c})}. \tag{A24}$$

The first equality follows from substituting the table entry for MPB_c^* in case (i) directly as well as substituting the implication of that table entry that $S_\infty(0) = \hat{S}_0(1 - MPB_c^*/\theta H)$. The second equality follows from $\lim_{\mathcal{R}_0 \uparrow \mathcal{R}_0'} MPB_c^* = c$ by continuity and from substituting from (24). Using the table entry for MSB_c^* in case (ii),

$$\lim_{\mathcal{R}_0 \downarrow \mathcal{R}_0'} MSB_c^* = \frac{\theta H \tilde{c}^2}{\tilde{c} + (1-\tilde{c})[\ln(1-\tilde{c}) + \mathcal{R}_0'\hat{I}_0]} = \frac{\theta H \tilde{c}(\hat{I}_0 + \tilde{c}\hat{S}_0)}{\hat{I}_0 + \tilde{c}\hat{S}_0 + (1-\tilde{c})\hat{S}_0 \ln(1-\tilde{c})}. \tag{A25}$$

The equality of (A24) and (A25) proves the continuity of MSB_c^* at $\mathcal{R}_0'$.

We next show MSB_c^* is quasiconcave for all $\mathcal{R}_0$ in case (i). Differentiating the relevant table entry, substituting from (A16), and eliminating positive constants shows that $\partial MSB_c^*/\partial \mathcal{R}_0$ has the same sign as

$$\left[\hat{S}_0 - S_\infty(0)\right]\left[1 - \mathcal{R}_0 S_\infty(0)\right] + \left[\hat{I}_0 + \hat{S}_0 - S_\infty(0)\right]\left(1 - \mathcal{R}_0\hat{S}_0\right). \tag{A26}$$

The second derivative of (A26) with respect to $\mathcal{R}_0$—after substituting from (A16), and rearranging considerably—can be shown to equal

$$2\frac{\partial S_\infty(0)}{\partial \mathcal{R}_0}\left[\hat{I}_0 + \hat{S}_0 - S_\infty(0)\right], \tag{A27}$$

which is negative—as can be shown using arguments similar to those behind Lemma 9. Hence, (A26) is concave. In the limit $\mathcal{R}_0 \downarrow 0$, (A26) approaches $2[\hat{S}_0 - S_\infty(0)] + \hat{I}_0$, which is positive by Lemma 4 and $\hat{I}_0 > 0$. Having established that (A26) is concave throughout (i) and initially positive, we have that (A26) can change sign at most once. Therefore, $\partial MSB_c^*/\partial \mathcal{R}_0$ is either nonnegative throughout case (i) or positive then negative. In either event, this proves that MSB_c^* is quasiconcave in (i). We have already established MSB_c^* is increasing in a neighborhood of $\mathcal{R}_0$ above 0, the last step needed to prove that MSB_c^* has a unique restricted local maximum over $\mathcal{R}_0 \le \mathcal{R}_0''$, which is a global maximum on that restricted set.

We next look at the behavior of MSB_c^* in cases (iii) and (iv), showing it has a most one restricted local maximum over $\mathcal{R}_0 > \mathcal{R}_0''$. Similar calculations used in the previous paragraph can be used here to establish the concavity of the following function,

$$\left[(1-\theta)\hat{S}_0-S_\infty(\hat{S}_0)\right]\left[1-\mathcal{R}_0S_\infty(\hat{S}_0)\right]+\left[\hat{I}_0+(1-\theta)\hat{S}_0-S_\infty(\hat{S}_0)\right]\left[1-(1-\theta)\mathcal{R}_0\hat{S}_0\right], \tag{A28}$$

which determines the sign of $\partial MSB_c^*/\partial\mathcal{R}_0$ in (iii) and (iv). Thus, (A28) has at most two roots in those cases, which cannot both be local maxima, implying that MSB_c^* has at most one local maximum over $\mathcal{R}_0>\mathcal{R}_0''$. The limit as $\mathcal{R}_0\uparrow\infty$ of (A28) equals

$$\hat{I}_0+2(1-\theta)\hat{S}_0-(1-\theta)\hat{S}_0[\hat{I}_0+(1-\theta)\hat{S}_0]\lim_{\mathcal{R}_0\uparrow\infty}\mathcal{R}_0 \tag{A29}$$

after substituting $\lim_{\mathcal{R}_0\uparrow\infty}S_\infty(\hat{S}_0)=\lim_{\mathcal{R}_0\uparrow\infty}[\mathcal{R}_0S_\infty(\hat{S}_0)]=0$ by Lemma 8. Expression (A29) approaches $-\infty$ since it involves $\mathcal{R}_0$ multiplied by negative constant. Since the limit $\mathcal{R}_0\uparrow\infty$ cannot produce a restricted supremum over $\mathcal{R}_0>\mathcal{R}_0''$, the restricted supremum is either the lower boundary of (iii), i.e., $\mathcal{R}_0''$, or is the interior restricted maximum. If $\mathcal{R}_0''$ provides the restricted supremum over $\mathcal{R}_0>\mathcal{R}_0''$, this cannot be a global maximum since MSB_c^* is decreasing in (ii); the restricted maximum over $\mathcal{R}_0\le\mathcal{R}_0''$ must then be the global maximum.

Results for MEX_c^*. We use the same roadmap for the comparative-statics analysis of MEX_c^* as for MSB_c^*. We begin by proving that MEX_c^* has a unique local restricted maximum over $\mathcal{R}_0\le\mathcal{R}_0''$. We do this by showing that MEX_c^* is increasing in a neighborhood around 0, quasiconcave for all $\mathcal{R}_0\in(0,\mathcal{R}_0']$, continuous at $\mathcal{R}_0'$, and decreasing for all $\mathcal{R}_0\in(\mathcal{R}_0',\mathcal{R}_0'')$. The arguments are made in reverse order. Differentiating the table entry for case (ii) yields

$$\frac{\partial MEX_c^*}{\partial\mathcal{R}_0}=\frac{-\theta H\tilde{c}^2(1-\tilde{c})\hat{I}_0}{\left\{\tilde{c}+(1-\tilde{c})[\ln(1-\tilde{c})+\mathcal{R}_0\hat{I}_0]\right\}^2}, \tag{A30}$$

which is negative. The proof that MEX_c^* is continuous at $\mathcal{R}_0'$ is similar to that for MSB_c^* and omitted.

We next show MEX_c^* is quasiconcave for all $\mathcal{R}_0$ in case (i). Differentiating the relevant table entry, substituting from equation (A16), and eliminating positive constants shows that $\partial MEX_c^*/\partial\mathcal{R}_0$ has the same sign as

$$\left[\hat{S}_0-S_\infty(0)\right]\left[1-\mathcal{R}_0(\hat{I}_0+\hat{S}_0)\right]+\mathcal{R}_0S_\infty(0)\left[1-\mathcal{R}_0S_\infty(0)\right]\left[\hat{I}_0+\hat{S}_0-S_\infty(0)\right]. \tag{A31}$$

All of the factors in (A31) are definitively positive except for $1-\mathcal{R}_0(\hat{I}_0+\hat{S}_0)$. If this is also nonnegative, then $\partial MEX_c^*/\partial\mathcal{R}_0$ is positive in (i), implying MEX_c^* is quasiconcave in (i), as desired.

So suppose instead that

$$\mathcal{R}_0(\hat{I}_0+\hat{S}_0)>1. \tag{A32}$$

We will show that (A32) implies that (A31) is concave. The second derivative of (A31) with respect to $\mathcal{R}_0$—after substituting from (A16), rearranging considerably, and removing positive factors—can be shown to have the same sign as

$$\begin{aligned}S_\infty(0)\left[1-\mathcal{R}_0(\hat{I}_0+\hat{S}_0)\right]-\left[1-\mathcal{R}_0S_\infty(0)\right](\hat{I}_0+\hat{S}_0)-S_\infty(0)\left[2(\hat{I}_0+\hat{S}_0)-S_\infty(0)\right]\\-S_\infty(0)\left\{1-\mathcal{R}_0\left[\hat{I}_0+\hat{S}_0-S_\infty(0)\right]\right\}\left[\hat{I}_0+\hat{S}_0-2S_\infty(0)\right].\end{aligned} \tag{A33}$$

By (A32) and familiar arguments, all the terms in (A33) are negative except possibly the last. If the last term is also nonpositive, the whole expression is negative, establishing (A31) is concave. So suppose instead that the last term is positive. For this to be the case, one of its last two factors must be positive and the other negative. That is, one of the following two sets of conditions must hold:

$$1-\mathcal{R}_0\left[\hat{I}_0+\hat{S}_0-S_\infty(0)\right]>0, \qquad \hat{I}_0+\hat{S}_0-2S_\infty(0)<0 \tag{A34}$$

$$1-\mathcal{R}_0\left[\hat{I}_0+\hat{S}_0-S_\infty(0)\right]<0, \qquad \hat{I}_0+\hat{S}_0-2S_\infty(0)>0 \tag{A35}$$

Suppose (A34) holds. Then (A33) is strictly less than

$$-S_\infty(0)\left[2(\hat{I}_0+\hat{S}_0)-S_\infty(0)\right]-S_\infty(0)\left\{1-\mathcal{R}_0\left[\hat{I}_0+\hat{S}_0-S_\infty(0)\right]\right\}\left[\hat{I}_0+\hat{S}_0-2S_\infty(0)\right] \tag{A36}$$

$$<-S_\infty(0)\left[2(\hat{I}_0+\hat{S}_0)-S_\infty(0)\right]-S_\infty(0)\left[\hat{I}_0+\hat{S}_0-2S_\infty(0)\right] \tag{A37}$$

$$=-3S_\infty(0)\left[\hat{I}_0+\hat{S}_0-S_\infty(0)\right]. \tag{A38}$$

Equation (A36) follows from eliminating the first two negative terms of (A33). Equation (A37) follows from substituting 1, which is greater than the factor in braces, for the factor in braces. The fact that this substitution results in an increase in (A37) follows from (A34). Straightforward algebra yields (A38), which is negative by familiar arguments.

Suppose (A35) holds. Then (A33) is strictly less than

$$S_\infty(0)\left[1-\mathcal{R}_0(\hat{I}_0+\hat{S}_0)\right]-S_\infty(0)\left\{1-\mathcal{R}_0\left[\hat{I}_0+\hat{S}_0-S_\infty(0)\right]\right\}\left[\hat{I}_0+\hat{S}_0-2S_\infty(0)\right] \tag{A39}$$

$$<S_\infty(0)\left[1-\mathcal{R}_0(\hat{I}_0+\hat{S}_0)\right]-S_\infty(0)\left\{1-\mathcal{R}_0\left[\hat{I}_0+\hat{S}_0-S_\infty(0)\right]\right\} \tag{A40}$$

$$=-\mathcal{R}_0 S_\infty(0)^2. \tag{A41}$$

Equation (A39) follows from eliminating the second and third two negative terms from (A33). Equation (A40) follows from substituting 1 for the last factor, $\hat{I}_0+\hat{S}_0-2S_\infty(0)$. To see that this increases the expression, note $\hat{I}_0+\hat{S}_0-2S_\infty(0)<\hat{I}_0+\hat{S}_0\leq 1$, where the last inequality holds since the size of the infected and susceptible subpopulations at date 0, $\hat{I}_0+\hat{S}_0$, cannot exceed the size of the entire population, normalized to 1. The fact that substituting 1 for $\hat{I}_0+\hat{S}_0-2S_\infty(0)$ increases (A39) follows from (A35). Straightforward algebra yields (A41), which is obviously negative.

In sum, we have shown (A33) is negative for $\mathcal{R}_0<\mathcal{R}_0'$, implying (A31) is concave. In the limit $\mathcal{R}_0\downarrow 0$, (A31) approaches $\hat{S}_0-S_\infty(0)$, which is positive by Lemma 4. These facts are sufficient to establish that MEX_c^* is quasiconcave in case (i) by the same arguments used for MSB_c^* above. These are all the facts needed to prove that MEX_c^* has a unique restricted local maximum over $\mathcal{R}_0\leq\mathcal{R}_0''$, which is a global maximum on that restricted set.

We next investigate the behavior of MEX_c^* in (iii) and (iv). Calculations similar to those used above can be used to show a function determining the sign of $\partial MEX_c^*/\partial\mathcal{R}_0$ in cases (iii) and (iv),

$$\left[(1-\theta)\hat{S}_0-S_\infty(0)\right]\left\{1-\mathcal{R}_0[\hat{I}_0+(1-\theta)\hat{S}_0]\right\}+\mathcal{R}_0 S_\infty(\hat{S}_0)\left[1-\mathcal{R}_0 S_\infty(\hat{S}_0)\right]\left[\hat{I}_0+(1-\theta)\hat{S}_0-S_\infty(\hat{S}_0)\right]. \tag{A42}$$

is concave. Thus, (A42) has at most two roots in cases (iii) and (iv), at most one of which is a local maximum for MEX_c^*. Taking the limit of the table entry for MEX_c^* in cases (iii) and (iv) and substituting the limit $\lim_{\mathcal{R}_0\uparrow\infty}S_\infty(\hat{S}_0)=\lim_{\mathcal{R}_0\uparrow\infty}[\mathcal{R}_0 S_\infty(\hat{S}_0)]=0$ by Lemma 8 yields $\lim_{\mathcal{R}_0\uparrow\infty}MEX_c^*=0$. Since the limit $\mathcal{R}_0\uparrow\infty$ produces an infimum for MEX_c^*, not a supremum, the restricted supremum of MEX_c^* over $\mathcal{R}_0>\mathcal{R}_0''$ is either the lower boundary of case (iii), i.e., $\mathcal{R}_0''$, or the interior restricted maximum. If $\mathcal{R}_0''$ provides the restricted supremum over $\mathcal{R}_0>\mathcal{R}_0''$, this cannot be a global maximum since MSB_c^* is decreasing in case (ii); the restricted maximum over $\mathcal{R}_0\leq\mathcal{R}_0''$ must then be the global maximum. *Q.E.D.*

Proof of Proposition 2

We first show that the condition for monopoly to deliver the first best involving universal vaccination of susceptibles, $MR(\hat{S}_0) \geq c$, holds for sufficiently high $\mathcal{R}_0$. Substituting $\hat{S}_0$ for Q in (26) shows that $MR(\hat{S}_0) \geq c$ holds if and only if

$$\Phi(\hat{S}_0)\left[1-\left(\frac{\theta}{1-\theta}\right)\frac{\mathcal{R}_0 S_\infty(\hat{S}_0)}{1-\mathcal{R}_0 S_\infty(\hat{S}_0)}\right] \geq \tilde{c}. \tag{A43}$$

The left-hand side equals 1 in the limit $\mathcal{R}_0 \uparrow \infty$. To see this, note that $\lim_{\mathcal{R}_0\uparrow\infty} S_\infty(Q) = 0$ by Lemma 8, implying $\lim_{\mathcal{R}_0\uparrow\infty} \Phi(\hat{S}_0) = 1$ by (14). Also by Lemma 8, $\lim_{\mathcal{R}_0\uparrow\infty} \mathcal{R}_0 S_\infty(Q) = 0$, implying the factor in square brackets in (A43) equals 1 in the limit $\mathcal{R}_0 \uparrow \infty$. The left-hand side exceeds the right-hand side in the limit since $1 > \tilde{c}$ by assumption (19).

We next verify that when monopoly output is an interior solution, i.e., $Q_m^* \in (0, \hat{S}_0)$, we have $Q_m^* < Q_c^*$. Given $Q_m^* > 0$, as argued in the text, $\mathcal{R}_0 > \mathcal{R}_0'$, ruling out case (i). As shown in Table 1 $Q_c^* = \hat{S}_0$ in cases (iii) and (iv), so it is immediate that $Q_m^* < Q_c^*$ for interior Q_m^*. This leaves case (ii). We have $MPB(Q_c^*) = MPB_c^* = c = MR(Q_m^*) < \theta H \Phi(Q_m^*) = MPB(Q_m^*)$, where the first step is definitional, the second step follows from the relevant entry in Table 1 in case (ii), the third step follows from the Kuhn-Tucker conditions for an interior solution, the fourth step follows from the fact that the factor in braces in (26) is less than 1, and the fifth step follows from (15). Since (16) is negative, $MPB(Q_c^*) < MPB(Q_m^*)$ implies $Q_c^* > Q_m^*$. *Q.E.D.*

Proof of Proposition 3

Results for Π_m^*. The result is a consequence of the Envelope Theorem. Monopoly profit can be written

$$\Pi_m^* = \theta H\left[1-\frac{S_\infty(Q_m^*)}{S_0(Q_m^*)}-\tilde{c}\right]Q_m^*. \tag{A44}$$

This is a function of $\mathcal{R}_0$ indirectly through its dependence on Q_m^*, which in turns depends on $\mathcal{R}_0$. It also depends on $\mathcal{R}_0$ because $S(Q)$ is a function of $\mathcal{R}_0$ (although the argument is omitted for brevity). If Q_m^* is an interior solution, as in case (ii) and (iii), the first-order condition ensures that the indirect effect of $\mathcal{R}_0$ on Π_m^* through Q_m^* equals 0. Only the direct effect remains. Hence,

$$\frac{\partial \Pi_m^*}{\partial \mathcal{R}_0} = \left[\frac{-\theta H Q_m^*}{S_0(Q_m^*)}\right]\frac{\partial S_\infty(Q_m^*)}{\partial \mathcal{R}_0}, \tag{A45}$$

which is positive since the derivative on the right-hand side is negative by Lemma 9.

Results for $R_\infty(Q_m^*)$. We first show that $R_\infty(Q_m^*)$ has at least one interior local maximum in $\mathcal{R}_0$. By Tables 1 and 2, $Q_m^* = Q_c^*$ in case (i), implying $R_\infty(Q_m^*) = R_\infty(Q_c^*)$. The proof of Proposition 1 showed $R_\infty(Q_c^*)$ is increasing in $\mathcal{R}_0$ in case (i), implying $R_\infty(Q_m^*)$ is increasing in case (i).

According to Table 2, $R_\infty(Q_m^*) = 1 - S_\infty(Q_m^*) - \theta Q_m^*$ in cases (ii) and (iii). Differentiating, substituting from (13), and rearranging yields

$$\frac{\partial R_\infty(Q_m^*)}{\partial \mathcal{R}_0} = -\left[\frac{\theta\Phi(Q_m^*)}{1-\mathcal{R}_0 S_\infty(Q_m^*)}\right]\frac{\partial Q_m^*}{\partial \mathcal{R}_0}. \tag{A46}$$

Since Q_m^* increases from 0 at the threshold $\mathcal{R}_0$ below case (ii) to $\hat{S}_0$ at the threshold above case (iii), we must have $\partial Q_m^*/\partial\mathcal{R}_0 > 0$ on a set of $\mathcal{R}_0$ in (ii) and (iii) of positive measure. Thus, $\partial R_\infty(Q_m^*)/\partial\mathcal{R}_0 < 0$ on a set of $\mathcal{R}_0$ of positive measure by (A46) since the factor in square brackets in (A46) is positive by Lemma 5. If $R_\infty(Q_m^*)$ is decreasing for $\mathcal{R}_0$ in a neighborhood above $\mathcal{R}_0'$ at the threshold between cases (i) and (ii), then $R_\infty(Q_m^*)$ attains a local maximum at $\mathcal{R}_0'$. Otherwise, the lower bound of the first set of positive measure for which $\partial R_\infty(Q_m^*)/\partial\mathcal{R}_0 < 0$ is a local maximum.

Suppose for the remainder of the proof that $\tilde{c} \geq 1-\theta$. We will show $R_\infty(Q_m^*)$ has an interior global maximum. Since $R_\infty(Q_m^*) = R_\infty(Q_c^*)$ for all $\mathcal{R}_0$ in case (i),

$$\lim_{\mathcal{R}_0\uparrow\mathcal{R}_0'} R_\infty(Q_m^*) = \lim_{\mathcal{R}_0\uparrow\mathcal{R}_0'} R_\infty(Q_c^*) = 1-(1-\tilde{c})\hat{S}_0, \tag{A47}$$

where the first equality follows by continuity since $\mathcal{R}_0'$ is the upper bound on case (i) by (24) and the second equality follows from (A21).

We proceed to compare (A47) to the limits of $R_\infty(Q_m^*)$ for extreme values of $\mathcal{R}_0$. We have

$$\lim_{\mathcal{R}_0\downarrow 0} R_\infty(Q_m^*) = \lim_{\mathcal{R}_0\downarrow 0} R_\infty(Q_c^*) = 1-\lim_{\mathcal{R}_0\downarrow 0} S_\infty(0) = 1-\hat{S}_0, \tag{A48}$$

where the first equality follows since $R_\infty(Q_m^*) = R_\infty(Q_c^*)$ for all $\mathcal{R}_0$ in case (i), the second equality follows from the entry for $R_\infty(Q_c^*)$ in case (i) in Table 1, and the third equality follows from Lemma 7. Equation (A48) is less than (A47). At the other extreme,

$$\lim_{\mathcal{R}_0\uparrow\infty} R_\infty(Q_m^*) = \lim_{\mathcal{R}_0\uparrow\infty} R_\infty(Q_c^*) = 1-\theta\hat{S}_0, \tag{A49}$$

where the first equality follows by continuity since $R_\infty(Q_m^*) = R_\infty(Q_c^*)$ for all $\mathcal{R}_0$ in case (iv), the second equality follows from the entry for $R_\infty(Q_c^*)$ in cases (iii) and (iv) in Table 1, and the third equality follows from (A23). Since $\tilde{c} \geq 1-\theta$, (A49) is weakly less than (A47). We have shown that $R_\infty(Q_m^*)$ is greater at the interior $\mathcal{R}_0'$ than at extreme values of $\mathcal{R}_0$, implying that $R_\infty(Q_m^*)$ has an interior global maximum.

Results for MSB_m^*. We show that the limits of MSB_m^* for extreme values of $\mathcal{R}_0$ are exceed by interior values. We have

$$\lim_{\mathcal{R}_0\downarrow 0} MSB_m^* = \lim_{\mathcal{R}_0\downarrow 0} MSB_c^* = \theta H\left[1-\frac{1}{\hat{S}_0}\lim_{\mathcal{R}_0\downarrow 0} S_\infty(0)\right] = \theta H\left(1-\frac{\hat{S}_0}{\hat{S}_0}\right) = 0, \tag{A50}$$

where the first equality follows since $MSB_m^* = MSB_c^*$ for all $\mathcal{R}_0$ in case (i), the second equality follows from the entry for MSB_c^* in case (i) in Table 1, and the third equality follows from Lemma 7. To examine the upper limit, the proof of Proposition 1 showed that MSB_c^* asymptotes downward toward $\lim_{\mathcal{R}_0\uparrow\infty} MSB_c^* = 1$. Since $MSB_m^* = MSB_c^*$ in case (iv), and all $\mathcal{R}_0$ above a sufficiently high value are contained in case (iv), MSB_m^* must also slope downward toward its asymptote. Thus MSB_m^* is higher at interior values of $\mathcal{R}_0$ than the extremes.

Results for MEX_m^*. Arguments similar to those just used for MSB_m^* can be used to show $\lim_{\mathcal{R}_0\downarrow 0} MEX_m^* = \lim_{\mathcal{R}_0\uparrow\infty} MEX_c^* = 0$. Hence, MEX_m^* is higher for interior values of $\mathcal{R}_0$ than extreme values and thus attains an interior maximum. *Q.E.D.*

Proof of Proposition 4

The sketch of the proof in the text omitted two details filled in here. We first prove Q^{**} for $\mathcal{R}_0$ in a neighborhood above 0. Taking limits in (22),

$$\lim_{\mathcal{R}_0 \downarrow 0} MSB(Q) = \theta H \left[1 - \frac{S_0(Q)}{S_0(Q)}\right] = 0. \tag{A51}$$

Hence, there exists $\mathcal{R}_0$ in a neighborhood above 0 and $\epsilon \in (0,c)$ such that $MSB(Q) < \epsilon$. For $\mathcal{R}_0$ in this neighborhood, $W(Q) = \int_0^Q [MSB(x) - c]dx < (\epsilon - c)Q < 0 = W(0)$. Thus, $Q^{**} = 0$ for $\mathcal{R}_0$ in this neighborhood.

We next prove $G^{**} > 0$ for some $\mathcal{R}_0 \in (0,\infty)$. Since $Q^{**} = 0$ for all $\mathcal{R}_0$ in neighborhood of 0, $Q^* \leq Q^{**} = 0$ implies $Q^* = 0$ for all $\mathcal{R}_0$ in a neighborhood of 0. The text argued $Q_m^* = \hat{S}_0$ for sufficiently high $\mathcal{R}_0$, implying $\hat{S}_0 = Q_m^* \leq Q^{**} \leq hS_0$, implying $Q^{**} = \hat{S}_0$ for sufficiently high $\mathcal{R}_0$. By the Theorem of the Maximum, since Q^{**} is a maximizer of continuous function $W(Q)$, Q^{**} is continuous, implying the existence of $\mathcal{R}_0 \in (0,\infty)$ such that $Q^* \in (0,\hat{S}_0)$. This Q^{**} must satisfy the first-order condition $MSB(Q^{**}) = c$, implying $MPB(Q^{**}) + MEX(Q^{**}) = c$, implying $MPB(Q^{**}) < c$ since $MEX(Q) > 0$ for all $Q \in (0,\hat{S}_0)$ by (23). *Q.E.D.*

Proof of Proposition 5

Start with the analysis of perfect competition. To derive G_c^{**} for various values of Q^{**}, first suppose $Q^{**} = 0$. Arguments in the text preceding Proposition 4 can be used to show $G_c^{**} = 0$.

Next, suppose $Q^{**} \in (0,\hat{S}_0)$. Then Q^{**} must satisfy the first-order condition for welfare maximization $MSB(Q^{**}) = c$, implying $MPB(Q^{**}) + MEX(Q^{**}) = c$, in turn implying $P_c^{**} = MPB(Q^{**}) = c - MEX(Q^{**})$. Since competitive firms pass the subsidy through to consumers, $P_c^{**} = c - G_c^{**}$. Combining the preceding equations yields $G_c^{**} = MEX(Q^{**})$.

Next, suppose $Q^{**} = \hat{S}_0 > Q_c^*$. Then the highest price at which output $\hat{S}_0$ is purchased satisfies $P_c^{**} = MPB(\hat{S}_0)$. Combined with competitive pass through, $P_c^{**} = c - G_c^{**}$, we have $G_c^{**} = c - MPB(\hat{S}_0)$.

Finally, suppose $Q_c^* = Q^{**} = \hat{S}_0$. Arguments in the text preceding Proposition 4 can be used to show $G_c^{**} = 0$. The various results for $Q^{**} = \hat{S}_0$ can be nested as $G_c^{**} = \max[0, c - MPB(\hat{S}_0)]$.

Turn next to the analysis of monopoly. To derive G_m^{**} for various values of Q^{**}, first suppose $Q^{**} = 0$. Arguments in the text preceding Proposition 4 can be used to show $G_m^{**} = 0$.

Next, suppose $Q^{**} \in (0,\hat{S}_0)$. The monopoly regards the subsidy as a reduction in marginal cost, maximizing $[MPB(Q) - c + G]Q$. To generate the first best, the optimal subsidy G_m^{**} must force the monopoly's first-order condition to be satisfied by Q^{**}:

$$MR(Q^{**}) = c - G_m^{**}. \tag{A52}$$

For general Q, (23) and (26) can be combined to show

$$MR(Q) = MPB(Q) - \frac{MEX(Q)\theta Q}{\hat{S}_0 - \theta Q}. \tag{A53}$$

Evaluting (A53) at $Q = Q^{**}$ yields

$$MR(Q^{**}) = MPB(Q^{**}) - \frac{MEX(Q^{**})\theta Q^{**}}{\hat{S}_0 - \theta Q^{**}} = c - MEX(Q^{**}) - \frac{MEX(Q^{**})\theta Q^{**}}{\hat{S}_0 - \theta Q^{**}}, \tag{A54}$$

where the second step follows from $MPB(Q^{**}) = c - MEX(Q^{**})$, which was shown in the analysis of perfect competition above when $Q^{**} \in (0, \hat{S}_0)$. Combining (A52) and (A54) and rearranging yields $G_m^{**} = MEX(Q^{**})\hat{S}_0/(\hat{S}_0 - \theta Q^{**})$.

Next, suppose $Q^{**} = \hat{S}_0 > Q_m^*$. According to standard Kuhn-Tucker conditions, for a subsidy G to induce the monopoly to produce at the corner $\hat{S}_0$, G must satisfy $MR(\hat{S}_0) \geq c - G$. This condition holds with equality at the lowest such subsidy, which is the optimal subsidy under monopoly, implying

$$MR(\hat{S}_0) = c - G_m^{**}. \tag{A55}$$

Evaluating (A53) at $Q = \hat{S}_0$ yields

$$MR(\hat{S}_0) = MPB(\hat{S}_0) - \frac{MEX(Q)\theta}{1-\theta}. \tag{A56}$$

Combining (A55) and (A56) yields

$$G_m^{**} = c - MPB(\hat{S}_0) + \left(\frac{\theta}{1-\theta}\right) MEX(\hat{S}_0). \tag{A57}$$

Finally, suppose $Q_c^* = Q^{**} = \hat{S}_0$. Arguments in the text preceding Proposition 4 can be used to show $G_m^{**} = 0$. The various results for $Q^{**} = \hat{S}_0$ can be nested as stated in (27). *Q.E.D.*

Proof of Proposition 6

Section 3.1 argued that universal vaccination is attained under perfect competition if $\mathcal{R}_0 > \mathcal{R}_0''$. Substituting $\theta = 1$ into the expression for $\mathcal{R}_0''$ in (25) yields $\mathcal{R}_0 > |\ln(1-\tilde{c})|/\hat{I}_0$. Rearranging and exponentiating yields $1 - e^{-\mathcal{R}_0 \hat{I}_0} > \tilde{c}$.

Turning next to the analysis of monopoly, according to Proposition 2, monopoly attains universal vaccination if and only if $MR(\hat{S}_0) \geq c$. Using (15) and (26) and rearranging, this inequality can be written

$$\Phi(\hat{S}_0)\left\{1 - \frac{\theta \mathcal{R}_0 \hat{S}_0 [1 - \Phi(\hat{S}_0)]}{1 - \mathcal{R}_0 S_\infty(\hat{S}_0)}\right\} \geq \tilde{c}. \tag{A58}$$

To determine whether (A58) holds with a perfectly effective vaccine, we need to take limits as $\theta \uparrow 1$, requiring us to compute limits $\lim_{\theta\uparrow 1} S_\infty(\hat{S}_0)$ and $\lim_{\theta\uparrow 1} \Phi(\hat{S}_0)$. To compute the first limit, by (12), $\lim_{\theta\uparrow 1} S_\infty(\hat{S}_0) = |\bar{L}(0)|/\mathcal{R}_0 = 0$, where the second step follows from the well-known fact that $\bar{L}(0) = 0$. Computing the second limit is more delicate since it involves a $0/0$ form. Manipulating (11), we have

$$\frac{S_\infty(Q)}{S_0(Q)} = e^{-\mathcal{R}_0[\hat{I}_0 + \hat{S}_0 - \theta Q - S_\infty(Q)]}, \tag{A59}$$

implying

$$\lim_{\theta\uparrow 1}\left[\frac{S_\infty(\hat{S}_0)}{S_0(\hat{S}_0)}\right] = e^{-\mathcal{R}_0 \hat{I}_0}, \tag{A60}$$

using $\lim_{\theta\uparrow 1} S_\infty(\hat{S}_0) = 0$. Hence, $\lim_{\theta\uparrow 1} \Phi(\hat{S}_0) = 1 - e^{-\mathcal{R}_0 \hat{I}_0}$. Substituting these limits into (A58) and recognizing that the inequality must be strict to hold for $\theta < 1$ yields $(1 - e^{-\mathcal{R}_0 \hat{I}_0})(1 - \mathcal{R}_0 \hat{S}_0 e^{-\mathcal{R}_0 \hat{I}_0}) > \tilde{c}$. *Q.E.D.*

Proof of Proposition 8

Suppose $\mathcal{R}_0\hat{S}_0 > 2$. Then $1 < \mathcal{R}_0\hat{S}_0/2 < \mathcal{R}_0[\hat{S}_0 + S_\infty(0)]/2 = \mathcal{R}_0[S_0(0) + S_\infty(0)]/2$, where the second step follows from $S_\infty(0) > 0$ by Lemma 4. This chain of inequalities implies that (29) holds at $Q = 0$ and thus that the vaccine exhibits initially increasing social returns.

At a general output level $Q \in (0, \hat{S}_0)$,

$$\mathcal{R}_0\left[\frac{S_0(Q) + S_\infty(Q)}{2}\right] > \mathcal{R}_0\left(\frac{S_0(Q)}{2}\right) = \mathcal{R}_0\left(\frac{\hat{S}_0 - \theta Q}{2}\right) \geq \mathcal{R}_0\left(\frac{(1-\theta)\hat{S}_0}{2}\right). \tag{A61}$$

If $\mathcal{R}_0\hat{S}_0 \geq 2/(1-\theta)$, then the last expression weakly exceeds 1, implying (29) holds for all feasible Q, implying the vaccine exhibits everywhere increasing social returns. *Q.E.D.*

Proof of Proposition 9

The assumption $c = 0$ implies $\tilde{c} = 0$, leaving two cases in Table 1: (ii)–(iii) and (iv). Nesting those cases, we can write

$$\Delta\Pi_m^* = \theta H\left\{\hat{I}_0 + \hat{S}_0\Phi(0) - Q_{mv}^*\Phi(Q_{mv}^*)\right\}, \tag{A62}$$

where Q_{mv}^* solves $\max_{Q\in[0,\hat{S}_0]} Q\Phi(Q)$. Since $Q_{mv}^* > 0$, we have $Q_{mv}^*\Phi(Q_{mv}^*) < Q_{mv}^*\Phi(0) \leq \hat{S}_0\Phi(0)$, where the first inquality follows from Lemma 10 and the second inequality from $Q_{mv}^* \in [0, \hat{S}_0]$. Substituting the preceding inequality into (A62) yields $\Delta\Pi_m^* > \theta H\hat{I}_0$. Thus, $\Delta\Pi_m^* > 0$ for all $\mathcal{R}_0 > 0$.

To derive the results on limits of $\Delta\Pi_m^*$, we have that $\lim_{\mathcal{R}_0\downarrow 0}\Phi(Q) = \lim_{\mathcal{R}_0\downarrow 0}[1 - S_\infty(Q)/S_0(Q)] = 1$ for all $Q \in [0, \hat{S}_0]$ since $\lim_{\mathcal{R}_0\downarrow 0} S_\infty(Q) = \hat{S}_0 - \theta Q = S_0(Q)$ by Lemma 7. Hence, $\lim_{\mathcal{R}_0\downarrow 0}\Delta\Pi_m^* = \theta H\hat{I}_0$. For all $Q \in [0, \hat{S}_0]$, $\lim_{\mathcal{R}_0\uparrow\infty}\Phi(Q) = 1$ since $\lim_{\mathcal{R}_0\uparrow\infty} S_\infty(Q) = 0$ by Lemma 8. Therefore,

$$\lim_{\mathcal{R}_0\uparrow\infty} Q_{mv}^*\Phi(Q_{mv}^*) = \lim_{\mathcal{R}_0\uparrow\infty}\left\{\max_{Q\in[0,\hat{S}_0]} Q\Phi(Q)\right\} = \max_{Q\in[0,\hat{S}_0]}\left[Q\lim_{\mathcal{R}_0\uparrow\infty}\Phi(Q)\right] = \hat{S}_0\cdot 1. \tag{A63}$$

Substituting from (A63) into (A62) along with $\lim_{\mathcal{R}_0\uparrow\infty}\Phi(0) = 1$ yields $\lim_{\mathcal{R}_0\uparrow\infty}\Delta\Pi_m^* = \theta H\hat{I}_0$. Now $\Delta\Pi_m^* > \theta H\hat{I}_0$ for all $\mathcal{R}_0 > 0$ implies $\theta H\hat{I}_0 \leq \inf_{\mathcal{R}_0>0}\Delta\Pi_m^* \leq \lim_{\mathcal{R}_0\downarrow 0}\Delta\Pi_m^* = \theta H\hat{I}_0$, which in turn implies $\inf_{\mathcal{R}_0>0}\Delta\Pi_m^* = \theta H\hat{I}_0$.

Combining the results from the previous paragraph, $\lim_{\mathcal{R}_0\downarrow 0}\Delta\Pi_m^* = \lim_{\mathcal{R}_0\uparrow\infty}\Delta\Pi_m^* = \inf_{\mathcal{R}_0>0} = \theta H\hat{I}_0$. But the first paragraph showed $\Delta\Pi_m^* > \theta H\hat{I}_0$. Hence, $\Delta\Pi_m^*$ must be nonmonotonic in $\mathcal{R}_0$, higher in the interior than for either limiting value of $\mathcal{R}_0$.

Turning to limiting values of ΔW_m^* as $\mathcal{R}_0 \downarrow 0$ and $\mathcal{R}_0 \uparrow \infty$, one can show that (A43) holds in these limits. Thus, the relevant case for computing W_{mv}^* is (iv). Substituting $\tilde{c} = 0$ into the relevant entry of Table 2 and multiplying by $\theta H\hat{S}_0$ to reverse the rescaling yields $W_{mv}^* = H[S_\infty(\hat{S}_0) + \theta\hat{S}_0]$. Subtracting from (31) and rearranging yields

$$\Delta W_m^* = H\left[\theta\hat{I}_0 + (1-\theta)S_\infty(0) - S_\infty(\hat{S}_0)\right]. \tag{A64}$$

By Lemma 7, $\lim_{\mathcal{R}_0\downarrow 0}[(1-\theta)S_\infty(0)] = (1-\theta)\hat{S}_0$. The lemma also implies $\lim_{\mathcal{R}_0\downarrow 0} S_\infty(\hat{S}_0) = (1-\theta)\hat{S}_0$. Substituting these limits into (A64) yields $\lim_{\mathcal{R}_0\downarrow 0}\Delta W_m^* = \theta H\hat{I}_0$. By Lemma 8, $\lim_{\mathcal{R}_0\uparrow\infty} S_\infty(0) = \lim_{\mathcal{R}_0\uparrow\infty} S_\infty(\hat{S}_0) = 0$. Substituting these limits into (A64) yields $\lim_{\mathcal{R}_0\uparrow 0}\Delta W_m^* = \theta H\hat{I}_0$.

The final step is to provide parameters for which $\Delta W_m^* < 0$. Using Matlab, we verified that for $\mathcal{R}_0 = 2$, $\theta = 0.5$, $\hat{I}_0 = 0.1$, $\hat{S}_0 = 0.8$, (A43) holds, implying that the vaccine monopoly supplies first-best quantity $\hat{S}_0$, putting us in case (iv). Subtracting the relevant Table 2 entry from (31) and simplifying yields $\Delta W_m^* = H[(1-\theta)S_\infty(0) + \theta\hat{I}_0 - S_\infty(\hat{S}_0)]$, which Matlab calculations show equals –0.09 for the specified parameters. *Q.E.D.*

Appendix A2. Calibration Details

The calibration considers the limiting case in which rescaled cost, $\tilde{c} = c/\theta H$, is set to 0. This appendix provides additional documentation justifying that limiting value.

Castillo et al. (2021) reports that prices for available Covid vaccines were no greater than \$40 per course. Health losses can be computed following Snyder et al. (2020). Hanlon et al. (2021) estimates 12 years of lost life (YLL) per death. Since this estimate already allocates shorter lifespans to people with comorbidities, we assume one YLL translates into one disability adjusted life year (DALY) without need for further downward adjustment to reflect a proportion of years lived with a disability. To convert DALYs into monetary values, we multiply DALYs lost in a country by three times that country's 2019 GDP per capita, reflecting World Health Organization (WHO) standards for a cost-effective health intervention in a country stated in Marseille et al. (2015). According to this standard, a health intervention is cost effective if the cost per DALY saved is less than three times that country's per-capita GDP (\$65,253 in the U.S. in 2019). Putting these estimates together yields an estimate of $H = 12 \times 3 \times \$65,253 = \$2.35$ million. Using the calibrated value of $\theta = 0.8$ yields $\tilde{c} = 40/(0.8 \times 2.35 \times 10^6) = 2.13 \times 10^{-5}$.

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Appendix A3. Cournot Competition

This appendix extends the analysis to Cournot competition, which nests the perfectly competitive and monopoly market structures studied in the text. Under Cournot competition, the vaccine is manufactured by $n \geq 1$ homogeneous firms, which choose quantities each period simultaneously. We will look for the symmetric Nash equilibrium, denoting a firm's equilibrium output by q_n^* and market output by $Q_n^* = nq_n^*$.

Case (i) from Table 1, in which $\mathcal{R}_0 \leq \mathcal{R}_0'$, which involves no sales under perfect competition, also involves no sales under Cournot since firms mark up marginal costs. Thus the entries in case (i) from both Tables 1 and 2 will also apply to Cournot. For the remainder of this appendix, suppose $\mathcal{R}_0 > \mathcal{R}_0'$.

Suppose market equilibrium output is an interior value: $Q_n^* \in (0, \hat{S}_0)$. Since some but not all consumers purchase, consumers must be indifferent between purchaing and not, implying the price must extract marginal private benefit: $P(Q) = MPB(Q)$. Thus, firm i's profit equals

$$[P(q_i + Q_{-i}) - c]q_i = [MPB(q_i + Q_{-i}) - c]q_i. \tag{A65}$$

Consider the following generalization of a firm's marginal revenue when n symmetric firms together produce Q units:

$$MR(Q,n) = MR(Q) = MPB(Q)\left\{1 - \frac{\theta \mathcal{R}_0 Q[1-\Phi(Q)]}{n[1-\mathcal{R}_0 S_\infty(Q)]}\right\}. \tag{A66}$$

The only difference from marginal revenue defined for a monopoly in (26) is the appearance of n in the denominator of the term in braces. It is obvious that (A66) reduces to (26) when $n = 1$. Taking the first-order condition of (A65) with respect to q_i, imposing symmetry by substituting $q_i^* = Q_n^*/n$, and rearranging, one can show that an interior equilibrium satisfies $MR(Q_n^*, n) = c$.

This interior solution is the equilibrium market output under Cournot if $Q_n^* < \hat{S}_0$. Otherwise, $Q_n^* = \hat{S}_0$, and all firms produce an equal share $q_n^* = \hat{S}_0/n$ in the symmetric equilibrium. A necessary and sufficient condition for this corner solution is $MR(\hat{S}_0, n) \geq c$.

Appendix A4. Consumer Heterogeneity

The model in the text assumes consumers are homogeneous. This appendix introduces consumer heterogeneity and shows that the key result regarding the nonmonotonicities of the marginal externality continues to hold in this extension.

For concreteness, assume consumers, indexed by i, differ in disease harm, H_i. Similar analysis applies if consumers experience different efficacies θ_i or have different lifespans. We conjecture that the analysis is also similar if consumers contract the disease at different rates, but modeling heterogeneity in that dimension requires delicacy to avoid changing the epidemiological process.

Denote the probability density function (pdf) by $f(H_i)$, the cumulative distribution function (cdf) by $F(H_i)$, and the complementary cdf by $\bar{F}(H_i) = 1 - F(H_i)$, and the expected value by $E(H_i) = \int_0^\infty H_i f(H_i) dH_i$. Assume H_i has full support on $(0, \infty)$.

Assume further that the population distribution of H_i is common knowledge but the specific realization of H_i is consumer i's private information. The model requires consumers to be aware of their heterogeneity (for example, differences in income leading to different willingnesses to pay to avoid harm, or a family history of disease). Undiagnosed conditions that lead harm to vary but are unknown to the consumer are better accommodated in the homogeneous-harm model.

With homogeneous consumers, we showed the marginal private benefit can be written $MPB(Q) = \theta H \Phi(Q)$, the product of efficacy, harm, and probability of contracting the disease. With consumer heterogeneity, consumer i's marginal private benefit becomes $MPB_i(Q) = \theta H_i \Phi(Q)$.

Incorporating heterogeneity in some of the normative measures requires additional work to keep track of the high-value consumers who end up purchasing. We have

$$SB(Q) = \left\{ \left[1 - \Phi(Q)\right] \int_0^{\hat{H}} H_i f(H_i) dH_i + \left[1 - \Phi(Q) - \theta \Phi(Q)\right] \int_{\hat{H}}^{\infty} H_i f(H_i) dH_i \right\} \hat{S}_0. \tag{A67}$$

The first integral reflects the expected health experienced by those whose harm is below the threshold $\hat{H}$ for purchase. With no vaccine to protect them, consumer i in this group obtains H_i with probability $1 - \Phi(Q)$. The second integral reflects the expected health experienced by those who purchase. Consumer i in this group obtains H_i if either they would not have been infected anyway (probability $1 - \Phi(Q)$) or would have been infected without a vaccine but receive the vaccine protection (probability $\theta \Phi(Q)$). The final factor $\hat{S}_0$ allows the per-consumer surplus given by the integrals to be scaled up to the population of potential consumers. Differentiating (A67) yields

$$MSB(Q) = \left\{ -\frac{\partial \Phi(Q)}{\partial Q} \left[E(H_i) - \theta \int_{\hat{H}}^{\infty} H_i f(H_i) dH_i \right] + \theta \Phi(Q) \hat{H} f(\hat{H}) \frac{\partial \hat{H}}{\partial Q} \right\} \hat{S}_0. \tag{A68}$$

To compute $\partial \hat{H} / \partial Q$, note threshold consumer type $\hat{H}$ is given as an implicit function of Q by $Q = \bar{F}(\hat{H}) \hat{S}_0$. Totally differentiating this identity with respect to Q and rearranging yields $\partial \hat{H} / \partial Q = 1 / f(\hat{H}) \hat{S}_0$. Substituting this derivative into (A68) shows that the last term equals $\theta \hat{H} \Phi(Q)$. This is the private benefit of the threshold consumer, equal to MPB^* when evaluated at the equilibrium Q^*. Subtracting to compute $MEX^* = MSB^* - MPB^*$ leaves just the first term of (A68), as stated in the following lemma.

Lemma 11. *In the model with heterogeneity in consumer harm H_i, the marginal externality in both the long- and short-run analyses equals*

$$MEX^* = -\frac{\partial \Phi(Q^*)}{\partial Q} \left[E(H_i) - \theta \int_{\hat{H}(Q^*)}^{\infty} H_i f(H_i) dH_i \right] \hat{S}_0. \tag{A69}$$

Intuitively, Lemma 11 says that the marginal externality is proportional to $-\partial \Phi(Q^*) / \partial Q$, the decline in the equilibrium probability of infection for an unvaccinated individual when one additional susceptible is vaccinated. The proof of the next proposition shows that that leading factor approaches 0 as $\mathcal{R}_0 \downarrow 0$ since a noninfectious disease presents no danger of infection in either analysis. The factor also approaches 0 as $\mathcal{R}_0 \uparrow \infty$ in both analyses since the individual will almost certainly contract the infinitely infectious disease in any event—from someone who was vaccinated but for whom the vaccine was ineffective if no one else. The remaining factors are obviously positive and finite for all $\mathcal{R}_0$. Thus, MEX^* approaches 0 for extreme values of $\mathcal{R}_0$, implying it peaks for an interior value of $\mathcal{R}_0 \in (0, \infty)$, as the following proposition states.

Proposition 10. *In the model with heterogeneity in consumer harm H_i, MEX^* peaks for an interior value of $\mathcal{R}_0 \in (0, \infty)$ under both perfect competition and monopoly.*

Proof. It remains to analyze the limits of $\partial\Phi(Q^*)/\partial Q$ as $\mathcal{R}_0 \downarrow 0$ and $\mathcal{R}_0 \uparrow \infty$, showing that the limits equal 0 for both market structures. Lemma 7 states $\lim_{\mathcal{R}_0 \downarrow 0}[\mathcal{R}_0 S_\infty(Q)] = 0$, implying $\lim_{\mathcal{R}_0 \downarrow 0} \partial\Phi(Q)/\partial Q = 0$ by (A17). Lemma 8 states $\lim_{\mathcal{R}_0 \uparrow \infty}[\mathcal{R}_0 S_\infty(Q)] = 0$, implying $\lim_{\mathcal{R}_0 \uparrow \infty} \partial\Phi(Q)/\partial Q = 0$ by (A17). These limits both hold for all Q, including $Q = Q_c^*$ and $Q = Q_m^*$. *Q.E.D.*

Appendix A5. Additional Preventive

The model in the text assumes that the vaccine is the only preventive technology available to consumers. In practice, consumers may pursue other preventive technologies instead of or in addition to vaccines, including social distancing. The extension in this appendix extends the model to allow for a second technology having efficacy δ against the disease. For simplicity, we initially derive results supposing the second technology is competitively supplied at a zero price, so that all consumers adopt it. We then show the results generalize to the case in which the price of the competitively supplied second product is nonnegative, as long as the price is sufficiently low.

Generalizing Model

We will generalize the model so that the successfully immunized compartment Z_0 now covers all successfully protected from the disease by any product. The epidemiological model remains the same as in the text, governed by equations (1)–(8). The only change is that Z_0 can have a more general form than in (9). To accommodate that more general form, we will rederive expressions for epidemiological outcomes as functions not of the vaccine quantity Q but of the proportion protected Z_0. We will write the susceptible, infected, and recovered compartments as $S_t(Z_0)$, $I_t(Z_0)$, and $R_t(Z_0)$, respectively, and write the probability of infection as

$$\Phi(Z_0) = 1 - \frac{S_\infty(Z_0)}{S_0(Z_0)} = 1 - \frac{S_\infty(Z_0)}{\hat{S}_0 - Z_0}. \tag{A70}$$

Since only an initial condition has changed, not one of the laws of motion, most of the epidemiological outcomes remain the same as in the text. In particular, Lemmas 1–5 remain unchanged substituting $S_t(Z_0)$ and $I_t(Z_0)$ for $S_t(Q)$ and $I_t(Q)$. Lemma 6 would remain unchanged had it been written in terms of $S_0(Q)$, but given it is written in terms of the exogenous parameters, the equations characterizing $S_\infty(Z_0)$, (11) and (12), need to be replaced by

$$\ln S_\infty(Z_0) - \mathcal{R}_0 S_\infty(Z_0) = \ln(\hat{S}_0 - Z_0) - \mathcal{R}_0(\hat{I}_0 + \hat{S}_0 - Z_0) \tag{A71}$$

$$S_\infty(Q) = \frac{1}{\mathcal{R}_0} \left| \bar{L}\left(-\mathcal{R}_0(\hat{S}_0 - Z_0) e^{-\mathcal{R}_0(\hat{I}_0 + \hat{S}_0 - Z_0)}\right) \right|. \tag{A72}$$

Applying the Implicit Function Theorem to (A71) yields

$$\frac{\partial S_\infty(Z_0)}{\partial Z_0} = \frac{S_\infty(Z_0)}{S_0(Z_0)} \left[\frac{\mathcal{R}_0 S_0(Z_0) - 1}{1 - \mathcal{R}_0 S_\infty(Z_0)}\right] \tag{A73}$$

$$\frac{\partial S_\infty(Z_0)}{\partial \mathcal{R}_0} = \frac{-S_\infty(Z_0)}{1 - \mathcal{R}_0 S_\infty(Z_0)} [\hat{I}_0 + S_0(Z_0) - S_\infty(Z_0)]. \tag{A74}$$

Note the similarity of these derivatives to their analogs, (13) and (A16), in the model without the second preventive.

Additional Preventive Freely Available

Assume that whether technologies protect an individual are independent draws across consumers and across technologies. Assume further that consumers do not learn about the success or failure of one technology before buying the other, so make a simultaneous buying decision in period $t = 0$. Given that all $\hat{S}_0$ initial susceptibles consume the second technology with efficacy δ and Q susceptibles buy the vaccine with efficacy θ, the number of initial susceptibles who are protected is

$$Z_0(Q) = \delta\hat{S}_0 + (1-\delta)\theta Q, \tag{A75}$$

replacing equation (9) in the epidemiological model.

We will continue the notational convention of writing variables related to demand, supply, and social welfare in terms of the quantity Q of the vaccine, still the product of key interest since it alone is the potential target of subsidy. The marginal private benefit from the vaccine is

$$MPB(Q) = (1-\delta)\theta H\Phi(Z_0(Q)), \tag{A76}$$

and its derivative is

$$\frac{\partial MPB(Q)}{\partial Q} = \frac{-(1-\delta)\theta\mathcal{R}_0 S_\infty(Z_0(Q))MPB(Z_0(Q))}{S_0(Z_0(Q))[1-\mathcal{R}_0 S_\infty(Z_0(Q))]}. \tag{A77}$$

The demand curve remains the same as in (18), substituting the following expression for demand when a subset purchase for (17):

$$d(P) = \frac{1}{(1-\delta)\theta}\left\{(1-\delta)\hat{S}_0 + \frac{(1-\delta)\theta H}{P}\left[\frac{1}{\mathcal{R}_0}\ln\left(1-\frac{P}{(1-\delta)\theta H}\right) + \hat{I}_0\right]\right\}. \tag{A78}$$

Regarding normative measures, social benefit becomes

$$SB(Q) = H[1-R_\infty(Z_0(Q))] = H[S_\infty(Z_0(Q)) + Z_0(Q)]. \tag{A79}$$

The expressions for welfare, marginal social benefit, and marginal externality in (21)–(23) remain unchanged since they were already written in sufficient generality.

On the supply side, redefine normalized unit cost as

$$\tilde{c} = \frac{c}{(1-\delta)\theta H} = \tilde{c}. \tag{A80}$$

Marginal revenue for a monopoly becomes

$$MR(Q) = MPB(Q)\left\{1-\frac{(1-\delta)\theta\mathcal{R}_0 Q[1-\Phi(Z_0(Q))]}{1-\mathcal{R}_0 S_\infty(Z_0(Q))}\right\}. \tag{A81}$$

The preceding expressions can be used to derive equilibrium variables under perfect competition and monopoly shown in Tables A1 and A2. The threshold values of $\mathcal{R}_0$ become

$$\mathcal{R}_0' = \frac{|\ln(1-\tilde{c})|}{\hat{I}_0 + (1-\delta)\tilde{c}\hat{S}_0}. \tag{A82}$$

$$\mathcal{R}_0'' = \frac{|\ln(1-\tilde{c})|}{\hat{I}_0 + (1-\delta)(1-\theta)\tilde{c}\hat{S}_0}. \tag{A83}$$

The entries in Table A1 can be used to show Proposition 1 holds without modification in this generalization. More specifically, for all variables except $R_\infty(Z_0(Q_c^*))$ and W_c^*, the entries are identical in the two tables after transforming two constants: $\breve{\theta} = (1-\delta)\theta$ and $\breve{S}_0 = (1-\delta)\hat{S}_0$. The entries for the remaining two variables just add a constant that does not affect the derivative with respect to $\mathcal{R}_0$, since $\partial S_\infty(Q)/\partial \mathcal{R}_0$ is invariant to the generalization as shown in (A74).

A similar argument can be used to show that Proposition 3 holds without modification in this generalization. It is immediate that Proposition 2 holds in the generalization because the expressions are provided in a general enough way that they remain unchanged in the generalization.

Turn next to an analysis of the comparative-statics effects of an increase in δ. The top panel of Figure A1 shows how Q^* varies with $\mathcal{R}_0$ for a given value of δ. An increase in δ effectively stretches the solid and dotted black curves for vaccine quantity rightward. Formally, one can show that an increase in δ increases the threshold values of $\mathcal{R}_0$ in (A82)–(A83) determining the regions in which some but not all consumers purchase a competitively supplied vaccines can be shown to increase in δ (taking into account the fact that an increase in δ increases $\tilde{c}$ in (A80)). The rightward stretch means that Q^* weakly declines in δ for a given $\mathcal{R}_0$. The reduction in vaccine quantity is not enough to offset the increase in population protection from the second preventive. Population protection increases in δ since each consumer has more options for protection, so can arrange weakly lower cost personal protection for any given level of population protection.

Additional Preventive Sold at Low Price

The top panel of Figure A1 illustrates the comparative-static effect of $\mathcal{R}_0$ on Q^* in the presence of a second preventive that is freely available. The picture is similar even when the second preventive is sold for a positive price if that price is sufficiently low that all consumers purchase the second preventive for any $\mathcal{R}_0$ such that any purchase the vaccine. The new situation is shown in the lower panel in Figure A1. While the gray curve representing the quantity of the second preventive looks different from the upper panel, it is only different in a region that is irrelevant for vaccine purchase; they are identical in the region of $\mathcal{R}_0$ labeled (d).

While it is intuitive that the strict results from the previous subsection, which hold when the price of the second preventive is zero, should hold in a neighborhood of strictly positive prices by continuity, we proceed to verify this formally. In particular, if the following condition holds,

$$c_2 < \frac{\delta c_1[1-\max(\delta,\theta)]}{(1-\delta)\theta}. \tag{A84}$$

then, for each $\mathcal{R}_0$ in regions (a)–(c) in Figure A1, there exists an equilibrium in which no vaccine is purchased and the quantity indicated purchase the second preventive; furthermore, for each $\mathcal{R}_0$ in region (d), there exists an equilibrium in which all consumers purchase the second preventive and the quantity indicated purchase the vaccine as well.

Case (a) is defined as that region of $\mathcal{R}_0$ for which no consumer purchases the second preventive when no vaccine is purchased either. For no consumer to purchase the second preventive in equilibrium, consumer surplus must be negative:

$$\delta H\Phi(0) - P_2 \leq 0. \tag{A85}$$

We will show that no consumer deviates to purchasing the vaccine either. We have

$$\theta H\Phi(0) - P_1 \leq \theta H\Phi(0) - c_1 \tag{A86}$$

$$< \theta H\Phi(0) - \frac{\theta c_2}{\delta} \tag{A87}$$

$$= \frac{\theta}{\delta}[\delta H\Phi(0) - c_2]. \tag{A88}$$

Condition (A86) follows from a nonnegative markup on vaccines, (A87) from $c_1 > \theta c_2/\delta$ by (A84), and (A88) from algebra. Substituting (A85) into (A88) implies $\theta H\Phi(0) - P_1 < 0$, implying that purchasing a vaccine provides negative consumer surplus for all $\mathcal{R}_0$ in case (a). The incremental consumer surplus from buying the vaccine in addition to the second preventive,

$$(1-\delta)\theta H\Phi(0) - P_1, \tag{A89}$$

is yet lower, so no vaccine is purchased in region (a).

In case (b), some but not all consumers purchase the second preventive, implying that the equilibrium quantity of the second preventive Q_2 is such that they are indifferent between buying the second preventive and not:

$$\delta H\Phi(\delta Q_2) - P_2 = 0. \tag{A90}$$

Conditions (A86)–(A88) continue to apply. Substituting (A90) into (A88) implies $\theta H\Phi(0) - P_1 < 0$, implying that purchasing a vaccine provides negative consumer surplus in case (b). As argued in the previous paragraph, the incremental consumer surplus from buying the vaccine in addition to the second preventive is yet lower, so no vaccine is purchased in region (b).

The upper threshold of case (c) is given by the $\mathcal{R}_0$ such that consumers first start to buy the vaccine in addition to the second preventive when all other consumers buy the second preventive and only that, when the vaccine is supplied under perfect competition. The marginal vaccine consumer obtains zero incremental surplus from a vaccine sold at marginal cost:

$$(1-\delta)\theta H\Phi(\delta\hat{S}_0) - c_1 = 0. \tag{A91}$$

We will show that consumers do not prefer buying just the vaccine to buying just the second preventive. The consumer surplus from buying just the vaccine is

$$\theta H\Phi(\delta\hat{S}_0) - P_1 \leq \theta H\Phi(\delta\hat{S}_0) - c_1 \tag{A92}$$

$$= \delta H\Phi(\delta\hat{S}_0) + (\theta-\delta)H\Phi(\delta\hat{S}_0) - c_1 \tag{A93}$$

$$= \delta H\Phi(\delta\hat{S}_0) - \frac{\delta(1-\theta)c_1}{(1-\delta)\theta} \tag{A94}$$

$$< \delta H\Phi(\delta\hat{S}_0) - c_2. \tag{A95}$$

Condition (A92) follows from a nonnegative markup on vaccines, (A87) from rearranging, (A94) from substituting from (A91) for $H\Phi(\delta\hat{S}_0)$ and rearranging, and (A95) from

$$c_1 > \frac{(1-\delta)\theta c_2}{\delta(1-\theta)}, \tag{A96}$$

which follows from (A84).

In case (d), consumers start to add the vaccine when all consumers purchase the second preventive. For each $\mathcal{R}_0$, there exists an equilibrium in which the analysis from the previous subsection characterizes equilibrium quantities of both products.

TABLE A1: Equilibrium Variables under Perfect Competition Adding Second Preventive Technology

	Case		
	(i)	(ii)	(iii), (iv)
Variable	$\mathcal{R}_0 \in (0, \mathcal{R}_0']$	$\mathcal{R}_0 \in (\mathcal{R}_0', \mathcal{R}_0'']$	$\mathcal{R}_0 \in (\mathcal{R}_0'', \infty)$
P_c^*	c	c	c
Q_c^*	0	$\frac{1}{(1-\delta)\theta}\left\{(1-\delta)\hat{S}_0 + \frac{1}{\tilde{c}}\left[\frac{1}{\mathcal{R}_0}\ln(1-\tilde{c}) + \hat{I}_0\right]\right\}$	$\hat{S}_0$
Π_c^*	0	0	0
$R_\infty(Z_0(Q_c^*))$	$1 - S_\infty(0) - \delta\hat{S}_0$	$1 - \hat{S}_0 - \hat{I}_0 + \frac{1}{\mathcal{R}_0}\lvert\ln(1-\tilde{c})\rvert$	$1 - S_\infty(\hat{S}_0) - [\delta + (1-\delta)\theta]\hat{S}_0$
MPB_c^*	$(1-\delta)\theta H\Phi(0)$	c	$(1-\delta)\theta H\Phi(\hat{S}_0)$
MSB_c^*	$\frac{(1-\delta)\theta H\Phi(0)}{1-\mathcal{R}_0 S_\infty(0)}$	$\frac{(1-\delta)\theta H\tilde{c}^2}{\tilde{c}+(1-\tilde{c})[\ln(1-\tilde{c})+\mathcal{R}_0\hat{I}_0]}$	$\frac{(1-\delta)\theta H\Phi(\hat{S}_0)}{1-\mathcal{R}_0 S_\infty(\hat{S}_0)}$
MEX_c^*	$\frac{(1-\delta)\theta H\Phi(0)\mathcal{R}_0 S_\infty(0)}{1-\mathcal{R}_0 S_\infty(0)}$	$\frac{(1-\delta)\theta H\tilde{c}(1-\tilde{c})[\lvert\ln(1-\tilde{c})\rvert-\mathcal{R}_0\hat{I}_0]}{\tilde{c}+(1-\tilde{c})[\ln(1-\tilde{c})+\mathcal{R}_0\hat{I}_0]}$	$\frac{(1-\delta)\theta H\Phi(\hat{S}_0)\mathcal{R}_0 S_\infty(\hat{S}_0)}{1-\mathcal{R}_0 S_\infty(\hat{S}_0)}$
W_c^*	$H[S_\infty(0) + \delta\hat{S}_0]$	$H[1 - (1-\delta)\tilde{c}]\hat{S}_0$	$H\{S_\infty(\hat{S}_0) + [\delta + (1-\delta)(1-\tilde{c})\theta]\hat{S}_0\}$

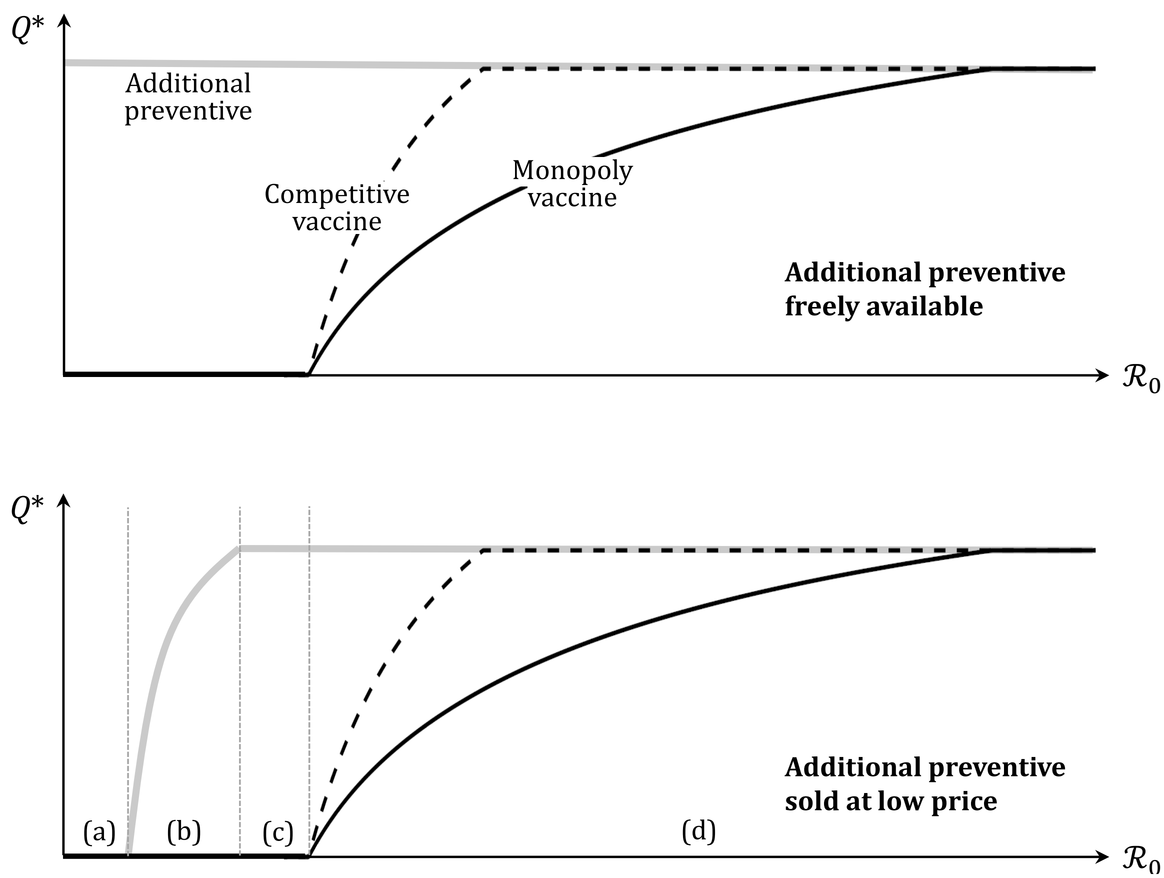
Notes: Generalizes Table 1, allowing for a second technology with efficacy δ against the disease. Entries in Table 1 can be recovered by setting $\delta = 0$. See that table for additional explanatory notes.

TABLE A2: Equilibrium Variables under Monopoly Adding Second Preventive Technology

	Case		
	(i)	(ii), (iii)	(iv)
Variable	$\mathcal{R}_0 \in (0, \mathcal{R}_0']$	$\mathcal{R}_0 > \mathcal{R}_0'$ but $MR(\hat{S}_0) < c$	$\mathcal{R}_0$ satisfies $MR(\hat{S}_0) \geq c$
P_m^*	†	$(1-\delta)\theta H\Phi(Q_m^*)$	$(1-\delta)\theta H\Phi(\hat{S}_0)$
Q_m^*	0	Solution to $MR(Q_m^*) = c$	$\hat{S}_0$
Π_m^*	0	$(1-\delta)\theta H[\Phi(Q_m^*) - \tilde{c}]Q_m^*$	$(1-\delta)\theta H[\Phi(\hat{S}_0) - \tilde{c}]\hat{S}_0$
$R_\infty(Z_0(Q_m^*))$	$1 - S_\infty(0) - \delta\hat{S}_0$	$1 - S_\infty(Q_m^*) - \delta\hat{S}_0 - (1-\delta)\theta Q_m^*$	$1 - S_\infty(\hat{S}_0) - [\delta + (1-\delta)\theta]\hat{S}_0$
MPB_m^*	$(1-\delta)\theta H\Phi(0)$	$(1-\delta)\theta H\Phi(Q_m^*)$	$(1-\delta)\theta H\Phi(\hat{S}_0)$
MSB_m^*	$\frac{(1-\delta)\theta H\Phi(0)}{1-\mathcal{R}_0 S_\infty(0)}$	$\frac{(1-\delta)\theta H\Phi(Q_m^*)}{1-\mathcal{R}_0 S_\infty(Q_m^*)}$	$\frac{(1-\delta)\theta H\Phi(\hat{S}_0)}{1-\mathcal{R}_0 S_\infty(\hat{S}_0)}$
MEX_m^*	$\frac{(1-\delta)\theta H\Phi(0)\mathcal{R}_0 S_\infty(0)}{1-\mathcal{R}_0 S_\infty(0)}$	$\frac{(1-\delta)\theta H\Phi(Q_m^*)\mathcal{R}_0 S_\infty(Q_m^*)}{1-\mathcal{R}_0 S_\infty(Q_m^*)}$	$\frac{(1-\delta)\theta H\Phi(\hat{S}_0)\mathcal{R}_0 S_\infty(\hat{S}_0)}{1-\mathcal{R}_0 S_\infty(\hat{S}_0)}$
W_m^*	$H[S_\infty(0) + \delta\hat{S}_0]$	$H[S_\infty(Q_m^*) + \delta\hat{S}_0 + (1-\delta)(1-\tilde{c})\theta Q_m^*]$	$H\{S_\infty(\hat{S}_0) + [\delta + (1-\delta)(1-\tilde{c})\theta]\hat{S}_0]$

Notes: Generalizes Table 2, allowing for a second technology with efficacy δ against the disease. Entries in Table 2 can be recovered by setting $\delta = 0$. See that table for additional explanatory notes.

FIGURE A1: Equilibrium Quantities at Various Prices for Second Preventive



Notes: Schematic diagram of comparative statics of Q^* in $\mathcal{R}_0$, analogous to top panel of Figure 1, but here for model with additional preventive. Panels illustrate two different prices for the second preventive. Dotted black curve represents Q_c^*, and solid black curve represents Q_m^*. Gray curve represents quantity of second preventive. Where curves overlap, solid black curve represents all overlapping curves.