Order-of-Magnitude Estimation
Falling Ball (Level 3)

The Question

How long does it take a pool ball dropped from a plane to hit the surface of the Earth?

Background

Many problems in physics can be reduced to "free fall" problems. The conditions for "free fall" are that an object falls under constant acceleration caused by the gravity due to a massive body. Constant acceleration is typically a reasonable assumption when the total amount of mass that is interior to the freely falling body does not change. Such conditions are typical for objects falling near the surface of the Earth.

Formally, the equations for acceleration under free fall conditions can be modeled by integrating the acceleration (a) and velocity (v) with respect to time (t) to determine how position (y) depends on acceleration. Typically, $v = 0$ at $t = 0$ (something is "dropped") and $y = 0$ at $t = 0$ (the reference frame is set up so that the initial position from which something is dropped is 0):

$$v = \int_0^t a\mathrm{d}t = at \quad , \quad y = \int_0^t v\mathrm{d}t = \int_0^t at\mathrm{d}t = \frac{1}{2}at^2 \tag{1}$$

A simplified, algebraic version of this derivation that is sufficient for OoM estimation is: speed = distance/time and acceleration = speed/time such that $a = v/t = (y/t)/t = y/t^2$.

The acceleration is the *effect* that gravity has on a falling object. The *cause* is the mass (M) of the body (of radius R) that the object (of mass m) is falling towards, which can be derived from Newton's Second Law and Newton's Law of Gravity:

$$F = ma = \frac{GMm}{R^2} \Rightarrow a = \frac{GM}{R^2} \tag{2}$$

where G is the Gravitational Constant.

Guiding Questions

Here are some things you may need to consider:

- Always have an initial guess at the answer without any OoM estimation!

- How can Newton's laws and the free fall equations be combined to derive time as a function of distance?
- How high does an airplane fly? In what units?
- Let's assume that we didn't know that the acceleration due to the Earth's gravity. How can we determine it?

The Solution

As we are not given the value of g, the acceleration due to the Earth's gravity, we first need to derive it. We can do so by equating weight from Newton's Second Law ($F = ma = mg$; where g is the acceleration due to the Earth's gravity) to the force of gravity from the Earth:

$$F = mg = \frac{GMm}{R^2} \Rightarrow g = \frac{GM}{R^2} \tag{3}$$

where R and M are the radius and the mass of the Earth, respectively.

How can we determine R? There are 4 time zones in the U.S. and the U.S. is about 4000 miles across, so there are about 1000 miles per time zone. There are 24 time zones around the Earth, so the Earth's circumference is about 24,000 miles. The radius of the Earth is therefore $24000/2\pi \sim 4000$ miles or ~ 6400 km, which is 6.4×10^6 m.

We can then determine:

$$g = \frac{GM}{R^2} = \frac{(6.7 \times 10^{-11})(6 \times 10^{24})}{(6.4 \times 10^6)^2} \sim \frac{4 \times 10^{13}}{4 \times 10^{12}} \sim 10\,\mathrm{m\,s^{-2}} \tag{4}$$

Note that we didn't have to calculate g explicitly, but it's a quite well-known number ($9.8\,\mathrm{m\,s^{-2}}$) so offers a good check on our calculation.

Now, we know the value of the free-fall acceleration due to the Earth's gravity, let's determine the time taken to fall a certain distance:

$$r = \frac{1}{2}at^2 = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2r}{g}} \tag{5}$$

How high does an airplane fly? About 30,000 feet. There are about 3 feet in a meter and so an airplane flies at about 10^4 meters. All that is left to do is to plug in the numbers:

$$t = \sqrt{\frac{2r}{g}} = \sqrt{\frac{2(10^4)}{10}} = \sqrt{2 \times 10^3} = \sqrt{20 \times 10^2} = \sqrt{20} \times \sqrt{10^2} \sim 4.5 \times 10^1 \sim 45\,\mathrm{s} \tag{6}$$

A ball in free-fall dropped from a plane takes about a minute to hit the ground.

Education Standards

This OoM Estimation problems meets the following standards in **bold**:
Next Generation Science Standards (NGSS):

- Physical Sciences
  - Matter & Its Interactions
  - **Motion and Stability: Forces and Interactions**
  - Energy
  - Waves and Their Applications in Technologies for Information Transfer
- Life Sciences
  - From Molecules to Organisms: Structures and Processes
  - **Ecosystems: Interactions, Energy, and Dynamics**
  - Heredity: Inheritance and Variation of Traits
  - Biological Evolution: Unity and Diversity
- Earth and Space Sciences
  - **Earth's Place in the Universe**
  - Earth's Systems
  - Earth and Human Activity
- Engineering, Technology, and Applications of Science
  - **Engineering Design**

Common Core Standards (CSS):

- **Counting & Cardinality**
- **Operations & Algebraic Thinking**
- **Numbers & Operations in Base Ten**
- **Number & Operations — Fractions**
- Measurement & Data
- **Geometry**
- **Ratios & Proportional Relationships**
- The Number System
- **Expressions & Equations**
- Functions
- Statistics & Probability