# QUANTUM DYNAMICS IN CONTINUOUS AND DISCRETE PHASE SPACE 

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## Abstract

This thesis presents new tools for understanding quantum dynamics on classical phase space. In the continuous-variable regime, a Wigner current for open quantum systems is constructed which leads to insights regarding the stabilization of Wigner function negative regions. The driven damped Duffing oscillator is used as a nonlinear quantum system in which to demonstrate the applications of the Wigner current. A similar current-based approach is then constructed for discrete-variable quantum systems in phase space. We find clear contrasts with probabilistic descriptions for discrete classical systems such as negativity arising in entities which play the quantum role of classical transition probabilities. This thesis will review the theoretical foundations upon which these ideas are assembled, thoroughly discuss how the dynamical descriptions are constructed, and present a number of applications.

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## Chapter 1

## Introduction

The discovery of quantum theory was a revolution in the understanding of our natural reality. Classical mechanics and determinism were supplanted with uncertainty, particle-wave duality, and probability. However, classical mechanics still provides a description that applies beautifully to many aspects of the world around us. As with the advent of any new theory, it must reproduce what proceeded it in the regimes that are appropriate. How does the classical world arise from its quantum underpinnings? What is the exact difference that distinguishes the two? Answering these questions is notoriously difficult and the subject of much ongoing research today.

In this thesis, we focus on developing tools with which quantum dynamics can be understood within the mathematical framework of phase space which is used to describe classical mechanics. We investigate both continuous- and discrete-variable quantum systems. In both cases, we find new ways to explain how quantum and classical behavior differ.

The theory of quantum mechanics is both highly successful and highly counterintuitive. Since its inception, the theory has yet to be contradicted by experiment. In some cases, as in the theory of quantum electrodynamics, predictions have been confirmed up to at least eight significant digits [102]. Quantum theory revolutionized

## Introduction

of understanding of the atomic world and those that could wield it controlled great power. The race for the atomic bomb is a sobering example of this. The impact of quantum theory is so great that it has driven the technological advances that have most dramatically shaped our society in the last hundred years. The invention of the transistor, leading then to the computer is the best example of a quantum-enabled technology that has transformed the face of our greater global society. Harnessing the stimulated emission of photons enabled the construction of the laser which finds applications in medicine, communications, industry, and a host of other fields. Quantum mechanics even enables our GPS systems which rely on atomic clocks.

Clearly, quantum mechanical effects at nanoscales can be leveraged very effectively. However, there is abundant motivation for exploiting quantum effects on scales that approach the macroscopic. Distinguishing between classical and quantum theories has become a practical issue in the fields of quantum information theory, quantum computing, and quantum metrology where an understanding of how the two theories differ can help with identifying where and how quantum protocols may outperform classical ones. It is therefore natural to employ a formulation of quantum mechanics that cleanly interfaces with the phase space theory of classical mechanics. The traditional formulation of the subject occurs in Hilbert space with quantum states represented by wavefunctions, or more generally as density operators. Because classical mechanics is formulated on phase space, it is difficult to compare similarities and contrast differences between the two theories. Recasting quantum theory onto phase space is one way that we can put the descriptions of the classical and the quantum on equal footing.

The Wigner function [142] is perhaps the most influential method of describing quantum states in phase space. It provides a particularly useful geometric representation of the state of a bosonic single mode quantum system as a real valued function
on the two-dimensional system phase space that can be interpreted as a quasiprobability density. In particular, integrating the Wigner function over any set of parallel lines in phase space yields a probability distribution of some observable that can be measured in the lab. In terms of the Wigner function, the quantum expectation value of a (Weyl ordered) observable $\hat{O}(q, p)$ is evaluated in exactly the same way as for the corresponding classical system described by a phase space probability distribution function. Furthermore, the master equation that describes the quantum dynamics of a bosonic single mode system interacting with its thermal environment gets mapped to a partial differential equation for the Wigner function dynamics that closely resembles the Fokker-Planck equation for the corresponding classical system statistical dynamics; the Wigner function dynamical equation may aptly be termed the 'quantum Fokker-Planck' equation. Given the close resemblance between the Wigner function representation of the open quantum system dynamical equations and the corresponding classical statistical dynamical equations, the Wigner function has helped provide an understanding of how classical dynamics arises by approximation from the underlying quantum dynamics [146, 79, 62, 96, 61, 41, 38, 57, 75, 129].

The current state of experimental progress makes this time particularly exciting to be exploring near-macroscopic quantum states as systems come online with particularly advantageous properties. Recent relevant developments in superconducting microwave resonator (as well as coupled nanomechanical resonator) circuits involving embedded Josephson junction elements provide strong motivation for pursuing such quantum states $[24,15,5,56,23,110,6,126,32]$; Josephson junctions can induce strong effective anharmonicities in the microwave mode Hamiltonian, as well as internally generated drive tones through the ac-Josephson effect. One consequence is lasing-like behavior [23], with the continuous, stimulated emission of amplitudesqueezed microwaves having large average photon number [5]. In addition, efforts to
combat the deleterious effects of noise have enabled quantum coherence that can now be maintained across many Josephson junction qubits. This was achieved with great fanfare in the recent demonstration of quantum supremacy [7]. The optomechanics community has also made significant steps forward towards realizing mesoscopic objects in the quantum regime $[16,101,131,8,25]$. In particular, levitated objects provide a particularly promising platform to investigate the quantum-to-classical transition due to relatively low decoherence and great parametric control over nonlinear Hamiltonian terms [36]. Wigner function descriptions can be extremely useful when studying the distinction between the quantum and classical behavior for these continuous- and discrete-variable systems.

In chapter 2 and 3 , we study continuous-variable quantum systems using the Wigner function. We review the formulation and important properties of the Wigner function in chapter 2. Special attention is paid to lay bare certain properties that will guide us when we turn our attention to finite-dimensional systems. In chapter 3, we look at how Wigner function dynamics can be understood in terms of currents. We extend the Wigner current construction to open quantum systems as reported in Braasch et al. [17]. That study was performed in collaboration with Oscar D. Friedman, Miles P. Blencowe, and Alexander J. Rimberg.

In chapters 4-6, we look at how the continuous variable approach to Wigner function dynamics may be extended to discrete variable systems. Chapter 4 provides a review of how the discrete Wigner function can be constructed with analogous properties to the continuous version $[143,138]$. We study closed system dynamics in chapter 5 where quantum behavior is detected via negativity in transition rates of quasiprobability between points in phase space. In chapter 6, we address general dynamical processes on discrete phase space such as those for open quantum systems. Again, negativity is found to be a uniquely quantum feature now in transition probabilities.

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The work reported in these chapters was performed in collaboration with William K. Wootters.

We conclude with chapter 7 , summarizing our results and looking forward to the natural extensions of the current state of our research.

## Chapter 2

## Quantum Mechanics in Phase

## Space

In this chapter and the next, we show how the theory of quantum mechanics can be constructed in phase space. Our motivation is to place quantum mechanics on the same mathematical foundation as classical mechanics in order to facilitate comparisons between their predictions. We focus on continuous-variable quantum systems that can be described with positions and momenta, similar to classical systems. We would expect quantum behavior to show up as a departure from how we understand the classical world in this space. This will make certain differences between classical and quantum behavior particularly apparent. Classical mechanics should appear in a certain limit of the quantum regime as we will be carefully detail. Finally, we will outline a few applications of the phase space formulation of quantum mechanics, with a bias towards our investigation of dynamics that follows in the next chapter.

Eugene Wigner introduced the most influential representation of a quantum state as a function on phase space [142], which is now known as the Wigner function. However, it is often hard for the uninitiated to grasp what elevates this function to such heightened importance amongst the milieu of other possibilities. The function that

Wigner wrote down appears in an ad hoc manner, as if it was pulled out of a hat that may have been shared with Leo Szilard. ${ }^{1}$ In this section we explore why the Wigner function is uniquely suited to a phase space description of quantum mechanics. Higher dimensional generalizations are straightforward, yet sacrifice clarity due to the demands of more cumbersome notation. Systems with a one-dimensional configuration space and a two-dimensional phase space will be sufficient for our purposes.

In standard quantum theory formulated in Hilbert space, probability distributions emerge from the dance between observables one might measure and quantum states as wavefunctions, or more generally density operators. Taking the modulus squared of the wavefunction $\psi(q)$ immediately gives us a probability distribution of the positions, in which upon measurement, we might find our system. The wavefunction in the momentum basis $\phi(p)$ is related to $\psi(q)$ via the Fourier transform. With $\phi(p)$, we can follow the same procedure of taking the modulus squared of the wavefunction to arrive at the momentum probability distribution.

A natural question one might ask is, can we form a joint probability distribution over position and momentum to represent a quantum state? Answering this question is directly in line with our goal to understand classical and quantum mechanics on the same mathematical foundation; thinking back to the power of Hamiltonian mechanics, one might be hopeful to leverage some of the same machinery in the quantum setting. In this description, the dynamics of particles are trajectories on phase space, however the idea of a localized trajectory seems at odds with a fundamental quantum mechanical principle. Due to the Heisenberg uncertainty principle, localizing the state of a quantum system to a point with well defined position and momentum in phase space is forbidden. This is a fundamental feature of quantum mechanics, but it is also just a feature found in any system with Fourier conjugate variables. Undeterred,

[^0]we will show that by relaxing our concept of legal probability distributions on phase space in a significant manner, we can construct a quantum theory on phase space. Furthermore, we will show how this perspective is not just a pleasant mathematical construct, but is quite utilitarian.

## - Section 2.1

## Quasiprobability Distribution Functions

A central task in quantum mechanics is the calculation of expectation values. Given a state described with a density operator $\hat{\rho}$, the expectation value of some operator $\hat{f}$ that is a function of position and momentum operators will be

$$
\begin{equation*}
\langle\hat{f}\rangle_{\text {Quantum }}=\operatorname{Tr}(\hat{\rho} \hat{f}) . \tag{2.1}
\end{equation*}
$$

Expectation values also play an important role in classical probability theory. In this context, a phase space probability distribution $P_{\text {Classical }}(q, p)$ enables the expectation value of a function $f(q, p)$ to be expressed as

$$
\begin{equation*}
\langle f\rangle_{\text {Classical }}=\int_{\mathbb{R}^{2}} \mathrm{~d} q \mathrm{~d} p f(q, p) P_{\text {Classical }}(q, p) \tag{2.2}
\end{equation*}
$$

Henceforth, all integrations will be taken over the total domain of the integrands unless otherwise indicated.

The classical and quantum expectation values are two very different entities and we would like to nail down exactly how they are different. A vague explanation that one might hear is that classical systems live in phase space while quantum systems live in Hilbert space. But this is unsatisfactory because our very intent is to understand how both might be formulated in phase space. In fact, there are at least three equivalent theories of quantum mechanics: the Hilbert space formulation, the
path-integral formulation, and the phase space formulation.
In order to investigate how one might draw a stark comparison between classical and quantum behavior, a formulation in phase space appears quite conducive. In this setting, certain differences between classical and quantum states and dynamics will be readily apparent. The most obvious difference that we will focus on is when we can interpret states represented as functions on phase as probability distributions. We must be careful to point out that this is not a panacea for all comparisons between classical and quantum physics on phase space. Analysis of the entire experiment including the state, transformations, and measurements must be confirmed to be classical [127].

Comparing the forms of Eqs. (2.1) and (2.2), it seems that a phase space treatment of quantum mechanics would entail representing density operators as probability distributions on phase space with observables as c-number functions. Given our original classical function $f(q, p)$, there is not a unique corresponding self-adjoint operator $\hat{f}(\hat{q}, \hat{p})$. We will discuss correspondence rules below, but for now it is enough to claim that such a correspondence can be made unique by choosing a distribution $P_{\text {Quantum }}$ that gives us an equation for the quantum expectation value that mimics that of the classical function:

$$
\begin{equation*}
\langle\hat{f}\rangle_{\text {Quantum }}=\int \mathrm{d} q \mathrm{~d} p f(q, p) P_{\text {Quantum }}(q, p) \tag{2.3}
\end{equation*}
$$

From a mathematical perspective, this statement is equivalent to saying that the density operator can determine a unique function on phase space. We will call such functions $P_{\text {Quantum }}$ quasiprobability distributions because while they act like probability distributions in many ways, there are some striking differences. What we must address is why this unique distribution cannot be a joint probability distribution.

The first quasiprobability distribution was introduced by Wigner [142] and is now
known as the Wigner function (or distribution). It will be denoted as $W(q, p)$. Other quasiprobability distributions include the $P$ function of Glauber [54] and Sudarshan [130] and the $Q$ function of Husimi [70, 73]. Due to a certain unique property of the Wigner function that we will describe below, this will be the quasiprobability distribution on which we focus. However, the other two distributions can be quite useful in certain applications, especially when working with electromagnetic fields as in quantum optics.

## Section 2.2

## Marginal Probabilities

As always, the place to start when making a model for the observable world is with measurable observations. In classical phase space, the probability of finding a particle in some arbitrary volume (or area in our case) in phase corresponds to the integration of the joint probability distribution over the arbitrary volume. We know we can't go this far for quantum systems due to the Heisenberg uncertainty principle which withholds us from measuring both our random variables, position and momentum, simultaneously. However, we can measure probabilities for finding the system with a specific position or momentum. The measurements that we know we can make on quantum systems will equate to measuring marginal probabilities if there exists a joint probability distribution. For a set of random variables, a marginal distribution is a probability distribution over a subset of the original random variables.

Are marginal probability distributions over only position and momentum enough to determine the joint distribution uniquely? The answer is "no" and we can come up with examples of many functions in phase space that reproduce the same marginals for $q$ and $p$. Start with a simple function on phase space that is given by the product of the marginal distributions $P(q)$ and $P(p): f(q, p)=P(q) P(p)$. We can alter this
function while retaining the correct marginals by adding any other function whose $q$ and $p$ marginals are both uniformly zero. Here is an interesting constraint determining a class of such functions:

$$
\begin{equation*}
f(q, p)=-f(-q, p)=-f(q,-p)=f(-q,-p) \tag{2.4}
\end{equation*}
$$

Clearly we need something stronger than just marginals for $q$ and $p$ to determine the exact distribution.

Quantum mechanical predictions allow us to build up probability distributions for not just the observables of position and momentum, but any observable that is a linear combination of the former. In the phase space setting, these other observables can be pictured as choosing a different angle for an axis over which to marginalize. Now we will make a short detour to prove the mathematical result that knowledge of the complete set of marginal probabilities is the same as knowledge of a distribution over the complete space. We will call this set of marginals a tomographically complete set. This discussion of marginals and their relation to the Wigner function closely follows that of Blass and Gurevich [14].

Let $f(q, p)$ be a joint or quasiprobability distribution on $\mathbb{R}^{2}$. We now create new observables $a q+b p$ out of linear combinations of position and momentum. The ratio of the real numbers $a$ and $b$ gives slopes on lines in phase space along which we can integrate to calculate marginal distributions. For every line $l=a q+b p$ where $a, b \in \mathbb{R}$ are not both zero, the marginal distribution $g(l)$ can be defined as such:

$$
g(l)= \begin{cases}\frac{1}{b} \int d q f\left(q, \frac{1}{b}(l-a q)\right) & \text { if } b \neq 0,  \tag{2.5}\\ \frac{1}{a} \int d p f\left(\frac{1}{a}(l-b p), p\right) & \text { otherwise }\end{cases}
$$

Here is an explanation for the case of $b \neq 0$. Rearranging the linear equation, we have

$$
\begin{align*}
p & =\frac{1}{b}(l-a q) \\
d p & =\frac{1}{b}(d l+a d q)  \tag{2.6}\\
f(q, p) d q d p & =f\left(q, \frac{1}{b}(l-a q)\right) \frac{1}{b} d q d l .
\end{align*}
$$

If we want to calculate the probability that a measurement of $a q+b p$ will give a value in some range $u \leq l \leq v$ with $u \leq v \in \mathbb{R}$, we find

$$
\begin{align*}
\int_{u}^{v} g(l) d l & =\iint_{u \leq a q+b p \leq v} d q d p f(q, p) \\
& =\iint_{u \leq a q+b p \leq v} d q d l \frac{1}{b} f\left(q, \frac{1}{b}(l-a q)\right)  \tag{2.7}\\
& =\int_{u}^{v} d l \int_{-\infty}^{\infty} d q \frac{1}{b} f\left(q, \frac{1}{b}(l-a q)\right)
\end{align*}
$$

Equating the integrands under $\int d l$ which hold for all $u \leq v$, we have

$$
\begin{equation*}
g(l)=\frac{1}{b} \int d q f\left(q, \frac{1}{b}(l-a q)\right) . \tag{2.8}
\end{equation*}
$$

The other case follows in a similar manner.
We now prove that for $g(l)$ to be a marginal distribution of $f(q, p)$ with $l=a q+b p$ is equivalent to the requirement that its Fourier transform satisfies the equation

$$
\begin{equation*}
\tilde{g}(\zeta)=\sqrt{2 \pi} \tilde{f}(a \zeta, b \zeta) \tag{2.9}
\end{equation*}
$$

Here $\zeta$ is a real number and $\tilde{f}$ is the double Fourier transform of $f: \tilde{f}(\xi, \eta)=$ $\frac{1}{2 \pi} \iint d q q p f(q, p) e^{-i(\xi q+\eta p)}$. The forward proof starts by assuming the initial state-
ment and calculating the Fourier transform of $g$ :

$$
\begin{align*}
\tilde{g}(\zeta) & =\frac{1}{\sqrt{2 \pi}} \int d l g(l) e^{-i \zeta l} \\
& =\frac{1}{\sqrt{2 \pi}} \iint d q d l \frac{1}{b} f\left(q, \frac{1}{b}(l-a q)\right) e^{-i \zeta l}  \tag{2.10}\\
& =\frac{1}{\sqrt{2 \pi}} \iint d q d p f(q, p) e^{-i \zeta(a q+b p)} \\
\tilde{f}(\xi, \eta) & =\frac{1}{2 \pi} \iint d q q p f(q, p) e^{-i(\xi q+\eta p)}
\end{align*}
$$

We used the form of $g$ derived above and we have $\tilde{g}(\zeta)=\sqrt{2 \pi} \tilde{f}(a \zeta, b \zeta)$. In the reverse direction of the statement, assume $\tilde{g}(\zeta)=\sqrt{2 \pi} \tilde{f}(a \zeta, b \zeta)$. If a function $h$ is a marginal distribution of $l=a q+b p$, we can use the forward implication to show that the Fourier transforms of both $h$ and $g$ have the same relationships with $f$. Because the Fourier transform is a bijection, we have that $g=h$ and we are done.

- Section 2.3


## The Wigner Function

Returning to the task of finding a phase space representation of quantum states, for some pure quantum state $|\psi\rangle$, suggestively let $W(q, p)$ be a function that yields all the correct marginal distributions $g(l)$ for linear combinations $l=a q+b p$ of position and momentum. For any real $\alpha, \beta, \zeta$ not all zero with $\alpha=a \zeta$ and $\beta=b \zeta$, we have from Eq. 2.9

$$
\begin{align*}
\tilde{W}(\alpha, \beta) & =\frac{1}{\sqrt{2 \pi}} \tilde{g}(\zeta)  \tag{2.11}\\
& =\frac{1}{2 \pi} \int d l g(l) e^{-i \zeta l}
\end{align*}
$$

Because $g(l)$ is a marginal probability distribution for $|\psi\rangle$, this is just an expectation value for the phase space observable $e^{-i \zeta l}$. Reformulating this as an expectation value

### 2.3 The Wigner Function

in Hilbert space for the Hilbert space observable $e^{-i \zeta(a \hat{q}+b \hat{p})}$ where $a \hat{q}+b \hat{p}$ is the Hilbert space observable associated with the line $l$, we have

$$
\begin{equation*}
\tilde{W}(\alpha, \beta)=\frac{1}{2 \pi}\langle\psi| e^{-i \zeta(a \hat{q}+b \hat{p})}|\psi\rangle . \tag{2.12}
\end{equation*}
$$

This last expression can be rewritten using the Baker-Campbell-Hausdorff formula and then inserting the completeness relation over positional states as

$$
\begin{align*}
\langle\psi| e^{-i \alpha \hat{q}-i \beta \hat{p}}|\psi\rangle & =e^{i \alpha \beta \hbar / 2}\langle\psi| e^{-i \alpha \hat{q}} e^{-i \beta \hat{q}}|\psi\rangle \\
& =e^{i \alpha \beta \hbar / 2} \int d y \psi^{*}(y) e^{-i \alpha y} \psi(y-\beta \hbar) . \tag{2.13}
\end{align*}
$$

We must now take an inverse Fourier transform to find

$$
\begin{equation*}
W(q, p)=\frac{1}{(2 \pi)^{2}} \iiint d y d \alpha d \beta e^{i \alpha \beta \hbar / 2} \psi^{*}(y) e^{-i \alpha y} \psi(y-\beta \hbar) e^{i \alpha q} e^{i \beta p} . \tag{2.14}
\end{equation*}
$$

The integration over $\alpha$ gives a Dirac delta function:

$$
\begin{equation*}
\int d \alpha e^{-i \alpha\left(y-\frac{\beta \hbar}{2}-q\right)}=2 \pi \delta\left(y-q-\frac{\beta \hbar}{2}\right) . \tag{2.15}
\end{equation*}
$$

A final integration over $y$ reveals the Wigner function

$$
\begin{equation*}
W(q, p)=\frac{1}{2 \pi} \int d \beta e^{i \beta p} \psi^{*}\left(q+\frac{\beta \hbar}{2}\right) \psi\left(q-\frac{\beta \hbar}{2}\right) . \tag{2.16}
\end{equation*}
$$

We can generalize this approach by considering an arbitrary density operator $\hat{\rho}=\sum_{j} p_{j}\left|\psi_{j}\right\rangle\left\langle\psi_{j}\right|$ where $j$ is the member of some index set and $p_{j}$ a probability. All the arguments go through for each $\left|\psi_{j}\right\rangle\left\langle\psi_{j}\right|$ with the new index on relevant elements
and the sum outside the equations. This leads to the more general expression

$$
\begin{equation*}
W(q, p)=\frac{1}{2 \pi} \int d \beta e^{i \beta p}\left\langle q+\frac{\beta \hbar}{2}\right| \hat{\rho}\left|q-\frac{\beta \hbar}{2}\right\rangle . \tag{2.17}
\end{equation*}
$$

Bras and kets containing the letter $q(p)$ are understood as eigenfunctions for the position operator $\hat{q}$ (momentum operator $\hat{p}$ ). This unique determination of the Wigner function from its marginal distributions can be derived in at least one other way that requires the function to satisfy Galilean covariance [12].

In the derivation above, we could have inserted a resolution of identity represented by momentum states and arrived at the momentum state representation:

$$
\begin{equation*}
W(q, p)=\frac{1}{2 \pi} \int d \alpha e^{-i \alpha q}\left\langle p+\frac{\alpha \hbar}{2}\right| \hat{\rho}\left|p-\frac{\alpha \hbar}{2}\right\rangle . \tag{2.18}
\end{equation*}
$$

We will shortly study the properties of this function and check to see if we can consider it as a probability distribution. Upon first sight, it is rather hard to understand from its functional form. The Wigner function appears to be the Fourier transform into momentum space of the off-diagonal elements of $\hat{\rho}$. Below we will give a physical interpretation of the function, but the fact focus on to for now is that it uniquely reproduces the correct quantum mechanical marginal probability distributions.

## Section 2.4

## Unique Characterization of the Wigner Function

The Wigner function is the unique representation of the density operator that satisfies the following three properties [12]:
(a) For all $\rho, W(q, p)$ is real.
(b) For all $\rho_{1}$ and $\rho_{2}$,

$$
\begin{equation*}
\operatorname{Tr}\left(\rho_{1} \rho_{2}\right)=2 \pi \int_{\mathbb{R}^{2}} d q d p W_{1}(q, p) W_{2}(q, p) \tag{2.19}
\end{equation*}
$$

(c) For all $\rho$, integrating $W(q, p)$ along the line $l=a q+b p$ in phase space yields the marginal probability that a measurement of the observable $a \hat{q}+b \hat{p}$ returns the result $l$.

The first property can be readily observed by taking the complex conjugate of Eq. (2.17) and changing the integration variable from $\beta$ to $-\beta$. Before we show the second property, we must discuss the Weyl transform and Weyl symbol. A Weyl transform maps an operator $\hat{A}$ in Hilbert space to a unique function $W_{A}$ in phase space called the Weyl symbol. For some operator $\hat{A}$, the Weyl transform $W_{A}$ is defined by

$$
\begin{equation*}
W_{A}(q, p)=\int d \beta e^{-i p \beta / \hbar}\langle q+\beta / 2| \hat{A}|q-\beta / 2\rangle . \tag{2.20}
\end{equation*}
$$

In this light, we can view the Wigner function as just a Weyl transform of the density operator that is normalized by a factor of $1 / 2 \pi \hbar$. By integrating over the product of two Weyl symbols over phase space, one can show

$$
\begin{equation*}
\operatorname{Tr}[\hat{A} \hat{B}]=\frac{1}{h} \iint d q d p W_{A}(q, p) W_{B}(q, p) \tag{2.21}
\end{equation*}
$$

The second property follows from the recognition that the Wigner function is just the normalized Weyl symbol of the density operator. Adherence to the third property is what motivated our initial definition of the Wigner function so we know this will be true.

## - Section 2.5

## Displaced Parity Operators

At this point, it is worthwhile to step back and formulate another way to understand the Wigner function. We will explore the interpretation of the Wigner function as the expectation value of the displaced parity operator following Royer [115]. This will also be an important bridge to the study of Wigner functions for finite dimensional quantum systems in chapter 4,5 , and 6 .

Let us work with pure states for the time being; generalization to density operators only requires a convex combination of pure states which can be performed at any point, so we save it to the end for ease of notation. Rescaling the variables $\alpha$ and $\beta$ in Eqs. (2.17) and (2.18), we can express the Wigner function as

$$
\begin{align*}
W(q, p) & =\frac{1}{\pi \hbar} \int d \beta e^{2 i p \beta / \hbar}\langle q+\beta \mid \psi\rangle\langle\psi \mid q-\beta\rangle  \tag{2.22}\\
& =\frac{1}{\pi \hbar} \int d \alpha e^{-2 i q \alpha / \hbar}\langle p+\alpha \mid \psi\rangle\langle\psi \mid p-\alpha\rangle . \tag{2.23}
\end{align*}
$$

Moyal provided another expression [97]:

$$
\begin{equation*}
W(q, p)=h^{-2} \iint d \alpha d \beta e^{-1(\alpha q+\beta p) / \hbar}\langle\psi| e^{i(\alpha \hat{q}+\beta \hat{p}) / \hbar}|\psi\rangle \tag{2.24}
\end{equation*}
$$

An astute statistician, Moyal noted that $\langle\psi| e^{i(\alpha \hat{q}+\beta \hat{p}) / \hbar}|\psi\rangle$ is a characteristic function that gives the expectation value of the operator $e^{i(\alpha \hat{q}+\beta \hat{p}) / \hbar}$. This operator corresponds to the function $e^{i(\alpha q+\beta p) / \hbar}$ via Weyl's correspondence principle. We will have more to say very shortly in this section about the correspondence between c-number phase space functions and operators in Hilbert space. We rewrite

$$
\begin{equation*}
W(q, p)=\frac{1}{\pi \hbar}\langle\psi| \hat{\Delta}(q, p)|\psi\rangle, \tag{2.25}
\end{equation*}
$$

where the operator $\hat{\Delta}(q, p)$ can take three equivalent forms making the connetion with Eqs. (2.22-2.24) explicit:

$$
\begin{align*}
\hat{\Delta}(q, p) & =\int d \beta e^{2 i p \beta / \hbar}|q-\beta\rangle\langle q+\beta|  \tag{2.26}\\
& =\int d \alpha e^{-2 i q \alpha / \hbar}|p+\alpha\rangle\langle p-\alpha|  \tag{2.27}\\
& =\frac{1}{4 \pi \hbar} \iint d \alpha d \beta \exp \left[\frac{i}{\hbar}(\alpha(\hat{q}-q)+\beta(\hat{p}-p))\right] . \tag{2.28}
\end{align*}
$$

For the case of $q=0, p=0$, we denote $\hat{\Delta}(0,0) \equiv \hat{\Delta}$ and have

$$
\begin{align*}
\hat{\Delta} & =\int d q|-q\rangle\langle q|  \tag{2.29}\\
& =\int d p|-p\rangle\langle p|  \tag{2.30}\\
& =\frac{1}{4 \pi \hbar} \iint d q d p \exp \left[\frac{i}{\hbar}(q \hat{q}+p \hat{p})\right] . \tag{2.31}
\end{align*}
$$

It is clear the $\hat{\Delta}$ is the parity operator centered at the origin changing the sign of variables in wavefunctions (e.g. $\hat{\Delta} \psi(q)=\psi(-q)$ ). The action on position and momentum operators is

$$
\begin{equation*}
\hat{\Delta} \hat{q} \hat{\Delta}=-\hat{q}, \quad \hat{\Delta} \hat{p} \hat{\Delta}=-\hat{p} \tag{2.32}
\end{equation*}
$$

Furthermore, with the displacement operators defined by

$$
\begin{equation*}
\hat{D}(q, p)=e^{-i q \hat{p} / \hbar} e^{i p \hat{q} / \hbar} \tag{2.33}
\end{equation*}
$$

we can connect the parity operator to what we shall now call displaced parity operators:

$$
\begin{equation*}
\hat{\Delta}(q, p)=\hat{D}(q, p) \hat{\Delta} \hat{D}^{\dagger}(q, p) \tag{2.34}
\end{equation*}
$$

The action of displacement operators on position and momentum operators is

$$
\begin{align*}
& \hat{D}(q, p)^{\dagger} \hat{q} \hat{D}(q, p)=\hat{q}+q  \tag{2.35}\\
& \hat{D}(q, p)^{\dagger} \hat{p} \hat{D}(q, p)=\hat{p}+p \tag{2.36}
\end{align*}
$$

Using Eqs. (2.32), (2.34), and (2.35) we find that

$$
\begin{align*}
& \hat{\Delta}(q, p)(\hat{q}-q) \hat{\Delta}(q, p)=-(\hat{q}-q)  \tag{2.37}\\
& \hat{\Delta}(q, p)(\hat{p}-p) \hat{\Delta}(q, p)=-(\hat{p}-p) \tag{2.38}
\end{align*}
$$

The displaced parity operator $\hat{\Delta}(q, p)$ reflects position and momentum coordinates about the point $(q, p)$. It also reveals in Eq. (2.25) the physical information that the Wigner function displays. As Royer remarks, the Wigner function is "a measure of how much $\psi$ is centered about $q, p$, and the Wigner distribution function now appears phyically more meaningful and natural than it did previously" [115]. Because $W$ is proportional to the overlap between $|\psi\rangle$ and its mirror image around $(q, p)$, he claims that this is a measure of how "centered" $|\psi\rangle$ is around that point. The Wigner function quantifies the reflection symmetry in position and momentum of a quantum state at a point.

The displaced parity operator is the pivotal technology we use to bridge the phasespace and Hilbert space formalisms of quantum mechanics. The three properties of the Wigner function can actually be understood as coming from three properties of the displaced parity operators. They are
(i) $\hat{\Delta}(q, p)$ is Hermitian.
(ii) $2 \pi \operatorname{Tr}\left[\hat{\Delta}(q, p) \hat{\Delta}\left(q^{\prime}, p^{\prime}\right)\right]=\delta\left(q-q^{\prime}\right) \delta\left(p-p^{\prime}\right)$.
(iii) For a projector $\Pi_{l}$ onto the eigenstate of $a \hat{q}+b \hat{p}$ with eigenvalue $l$,

$$
\begin{equation*}
\int_{\mathbb{R}^{2}} d q d p \hat{\Delta}(q, p) \delta(a q+b p-l)=\Pi_{l} \tag{2.39}
\end{equation*}
$$

Property (i) of the displaced parity operators is self-explanatory and straightforward to check. Property (ii) states that they form a basis for Hilbert-Schmidt trace class operators [34].

We can now write the expression for the Weyl symbol simply as

$$
\begin{align*}
W_{\hat{O}}(q, p) & =\operatorname{Tr}[\hat{O} \hat{\Delta}(q, p)]  \tag{2.40}\\
& =\int d y e^{i p y / \hbar}\langle q-y / 2| \hat{O}|q+y / 2\rangle . \tag{2.41}
\end{align*}
$$

For Hermitian operators, the Weyl symbols will always be real. Eq. (2.40) establishes a bijection between phase-space and Hilbert space with the inverse given by

$$
\begin{equation*}
\hat{O}=\frac{1}{2 \pi \hbar} \iint d q d p W_{\hat{O}}(q, p) \hat{\Delta}(q, p) \tag{2.42}
\end{equation*}
$$

The Wigner function is the Weyl symbol of the density operator $\hat{\rho}$ multiplied by the factor $\frac{1}{2 \pi \hbar}$ in order to enforce normalization.

- Section 2.6


## Weyl Symbols and Star-products

We have already understood the Wigner function as a Weyl symbol and stated that the Weyl transformation works for any Hilbert space operators. One natural area to investigate is the mapping of a product of two operators. There is a lot of interesting physics described mathematically by the lack of commutivity between certain operators. How can we imbue a c-number function on phase space with this non-
commutative property? Groenewold elegantly described the Weyl symbol obtained from the product of two operators using the $\star$-product ("star-product") between their separate Weyl symbols by

$$
\begin{equation*}
W_{\hat{A} \hat{B}}(q, p)=\left(W_{\hat{A}} \star W_{\hat{B}}\right)(q, p) \tag{2.43}
\end{equation*}
$$

where

$$
\begin{equation*}
\star \equiv e^{\frac{i \hbar}{2}\left(\overleftarrow{\partial}_{q} \vec{\partial}_{p}-\overleftarrow{\partial}_{p} \vec{\partial}_{q}\right)} \tag{2.44}
\end{equation*}
$$

We have used the shorthand notation: $\partial q \equiv \frac{\partial}{\partial q}$, and $\partial p \equiv \frac{\partial}{\partial p}$.
There are a number of equivalent representations. A Fourier representation is the generalized convolution [9]

$$
\begin{align*}
W_{\hat{A}} \star W_{\hat{B}}= & \frac{1}{\hbar^{2} \pi^{2}} \int d q^{\prime} d q^{\prime \prime} d p^{\prime} d p^{\prime \prime} W_{\hat{A}}\left(q^{\prime}, p^{\prime}\right) W_{\hat{B}}\left(q^{\prime \prime}, p^{\prime \prime}\right) \\
& \times \exp \left(\frac{-2 i}{\hbar}\left(p\left(q^{\prime}-q^{\prime \prime}\right)+p^{\prime}\left(q^{\prime \prime}-q\right)+p^{\prime \prime}\left(q-q^{\prime}\right)\right)\right) . \tag{2.45}
\end{align*}
$$

A discrete analog of this form will show up in our discussion of discrete Wigner functions later. A curiosity is that the argument of the exponent is twice the oriented area of a triangle in phase space with vertices at the respective primed pairs of coordinates multiplied by $-2 i / \hbar$. We will find a similar property in our study of Wigner functions in discrete phase space. For now, we will just mention that area plays an important role in both continuous and discrete phase space because it allows for quantification of the angle between two vectors. This can be seen by calculating the determinant of a matrix formed from a pair of two-dimensional vectors in phase space. This determinant collapses to zero as the vectors become parallel and is a measure of the area determined by the two vectors. The determinant is related to what is called the symplectic structure of phase space. We will discuss the discrete version of Eq. (2.45) in depth in chapter 4.

Another integral representation readily shows the symplectic nature of phase space [65]:

$$
\begin{align*}
W_{\hat{A}} \star W_{\hat{B}}= & (\hbar \pi)^{-2} \int d q^{\prime} d q^{\prime \prime} d p^{\prime} d p^{\prime \prime} W_{\hat{A}}\left(q+q^{\prime}, p+p^{\prime}\right) W_{\hat{B}}\left(q+q^{\prime \prime}, p+p^{\prime \prime}\right)  \tag{2.46}\\
& \times \exp \left(\frac{2 i}{\hbar}\left(q^{\prime} p^{\prime \prime}-q^{\prime \prime} p^{\prime}\right)\right)
\end{align*}
$$

Here we can see the associativity and noncommutativity on display.
Translation of functional arguments or "Bopp Shifts" enable a final representation:

$$
\begin{align*}
W_{\hat{A}} \star W_{\hat{B}} & =W_{\hat{A}}\left(q+\frac{i \hbar \vec{\partial}_{p}}{2}, p-\frac{i \hbar \vec{\partial}_{q}}{2}\right) W_{\hat{B}}(q, p) \\
& =W_{\hat{A}}(q, p) W_{\hat{B}}\left(q-\frac{i \hbar \overleftarrow{\partial}_{p}}{2}, p+\frac{i \hbar \overleftarrow{\partial}_{q}}{2}\right) \tag{2.47}
\end{align*}
$$

The arrows above the differential operators indicate that they act only on the other Weyl symbol. These equations with Bopp shifts can be quite useful in practice, such as when deriving the quantum Liouville equation in the next chapter.

The simplest examples of Weyl symbols are those of the position and momentum operators,

$$
\begin{equation*}
W_{\hat{q}}(q, p)=q \text { and } W_{\hat{p}}(q, p)=p . \tag{2.48}
\end{equation*}
$$

This correspondence with the classical phase space functions holds for any Weylordered moments

$$
\begin{equation*}
W_{\left\{\hat{p}^{k},,^{l}\right\}_{W}}(q, p)=p^{k} q^{l}, \tag{2.49}
\end{equation*}
$$

where the Weyl ordering is defined as

$$
\begin{equation*}
\left\{\hat{p}^{k}, \hat{q}^{l}\right\}_{W}=2^{-k} \sum_{j=0}^{k}\binom{k}{j} \hat{p}^{k-j} \hat{q}^{l} \hat{p}^{j} . \tag{2.50}
\end{equation*}
$$

Finally, it is evident that the Weyl symbol for the Hamiltonian $\hat{H}=\hat{p}^{2} / 2 m+V(\hat{q})$ is given by

$$
\begin{equation*}
W_{\hat{H}}(q, p)=\frac{p^{2}}{2 m}+V(q) \tag{2.51}
\end{equation*}
$$

Although the Wigner function has a cryptic form, the Weyl symbols for many simple operators coincide with the classical expressions.

## Section 2.7

## Wigner Function Geometry

We have carefully thought through the uniqueness of the Wigner function as a representation of continuous-variable quantum systems on phase space. In this section we show that the Wigner function must take negative values for certain quantum states. We will show how negativity arises and why negativity can be a sign of nonclassical behavior. Then we will investigate some constraints on negative regions.

The inner product rule given in Eq. (2.19) has two interesting consequences. We now focus only on pure states. If we set $|\psi\rangle=|\phi\rangle$, we have

$$
\begin{equation*}
\iint d q d p\left[W_{\psi}(q, p)\right]^{2}=\frac{1}{2 \pi \hbar} . \tag{2.52}
\end{equation*}
$$

Generalization of Eq. (2.52) to the case of a mixed state $\hat{\rho}=\sum_{j} p_{j}\left|\psi_{j}\right\rangle\left\langle\psi_{j}\right|$ with probabilities $p_{j}$ results in the right-hand side being multiplied by $\sum_{j} p_{j}^{2}$. Therefore classical-like Wigner functions $W(q, p)=\delta\left(q-q^{\prime}\right) \delta\left(p-p^{\prime}\right)$ with $q^{\prime}, p^{\prime}$ fixed are impossible because those states can never satisfy Eq. (2.52). On the other hand, if we choose $|\psi\rangle$ and $|\phi\rangle$ such that they are othogonal, it must be that

$$
\begin{equation*}
\iint d q d p W_{\psi}(q, p) W_{\phi}(q, p)=0 \tag{2.53}
\end{equation*}
$$

If we imagine that one of the states is strictly positive everywhere in phase space this puts us in a peculiar situation. Let $W_{\psi}$ be such a state. For Eq. (2.53) to hold, $W_{\phi}$ must have some regions where it takes negative values. In order to construct a distribution on phase space that behaves like a probability in the ways that are experimentally accessible to us, i.e. along marginal distributions, we will have to accept that our distribution may have negativity. This is why we will call the Wigner distribution function a quasiprobability.

We know that classical joint probability distributions do not allow negative values by their very definition. Does this mean that the presence of negativity in the Wigner function is a signature of nonclassicality? Alternatively, is a lack of negativity a signature of non-quantumness? Hudson showed that the quantum states that are considered the most classical have Wigner functions that do not have negative values [69]. These states are the coherent and squeezed states whose functional form is of a Gaussian in phase space. With the exception of Gaussian (i.e., coherent and squeezed) states, all pure states have negative-valued Wigner function regions and hence are nonclassical [69]. Well known examples are the harmonic oscillator energy eigenstates or Fock states and so-called Schrödinger cat states involving superpositions of different coherent states.

Coherent states and their squeezed siblings minimize the uncertainty principle and therefore are the most localized any quantum states can be in phase space. Coherent states minimize this uncertainty with complete angular symmetry around their centers (for a particular choice of the scale ratio between position and momentum). In this way, out of all quantum states, they most closely resemble the pointilist nature of classical particles that in principle may be localized to a specific point in phase space. Furthermore, as we will see in the next chapter, coherent states closely reproduce the classical dynamics of the particle in a harmonic potential in the closest possible fash-
ion. One can therefore use negativity as an indicator of nonclassicality. Going a step farther from indicator to measuring stick, the volume of a negative region calculated as the integral of the Wigner function over the region can be used as a measure of nonclassicality [76]. In support of using negativity to indicate non-classical behavior, the presence of a thermal environment with usually force nonclassical, pure states to evolve into mixed states with Wigner function representations that are everywhere positive [146].

Finally, we will briefly point out contributions to the issue of interpreting negativity in the Wigner function. We emphasize that any discomfort we have with these negative values stems just from our interpretation of the formalism and not any issue with the mathematics. The idea of considering the negative regions of the Wigner function has been advocated by Bartlett [10] and Feynman [44]. Feynman reminds us about how useful nonsensical negative quantities are in our description of the world; it only makes sense to talk of a negative integer number of apples if we are describing an intermediate state of affairs where one person is owed apples by someone else. There have also been efforts to interpret probability flows of negative regions as negative probability backflows [18, 98].

In conclusion, we have succeeded in formulating quantum mechanics on phase space. The Wigner function was shown to be the unique real function in phase space that correctly reproduces the marginal probabilities to which we have experimental access. We saw that when Wigner functions did not look like classical probability distributions, this was in fact a signature of quantum behavior. Now we are ready to think carefully about how Wigner function negativity can grow and disappear as we wish to coax its presence into the macroscopic world.

## Chapter 3

## Wigner Current

Up until this point, our discussions have been restricted to quantum states and devoid of time. We will now breathe some life into our quantum states and explore the Moyal equation - the dynamical equation for Wigner functions. Just as the Wigner function was shown to have certain forms that we considered classical, we will also find a classical regime of the Moyal equation. It will be shown that this classicality occurs only for harmonic systems while anharmonic nonlinear systems exhibit quantum corrections to the classical case in their dynamics.

Our key motivation for introducing the Wigner current and casting the bosonic mode system quantum dynamics into a continuity equation on phase space is to apply a more geometric approach to addressing the sought-after goal to generate macroscopic quantum states that are stable over long times against the decohering effects of the environment. By 'macroscopic', we mean that the averaged number of energy quanta (e.g., photons, phonons etc.) is large in the stabilized, bosonic mode state, while by 'quantum' we mean that the Wigner function representation of the state has significant negative regions in the system phase space.

Approaches to stabilizing quantum states involve measurement feedback to control the quantum system dynamics [71], as well as so-called autonomous methods that do
not require measurement feedback control. The latter typically involve 'reservoir engineering', where the effective system-environment interaction is tailored in such a way as to evolve the system into a quantum state as well as to protect the state from the decohering effects of the environment $[108,117,111,87,125,114,82,132]$.

Another approach to autonomously generating quantum states exploits the nonlinearities in the closed bosonic mode system dynamics-equivalently anharmonicities in the system Hamiltonian. The presence of anharmonicities can cause initial Gaussian states with associated positive Wigner functions to evolve into nonclassical states with associated negative valued Wigner functions (see, e.g., Ref. [75]). In terms of the quantum Fokker-Planck dynamical equations for the Wigner function, the root cause of such evolution is the presence of a third or higher order position derivative term involving the system potential energy. Only when the potential energy is anharmonic is this term present and without this term, the Wigner function dynamical equation coincides with the classical Fokker-Planck equation.

For example, in the case of the driven, damped Duffing oscillator with $q^{4}$ anharmonicity and in the regime of bistable large and small amplitude oscillatory solutions for the classical dynamics, an initial coherent state will transiently evolve into a Schrödinger cat-like state where the Wigner function displays a sequence of alternating negative and positive regions in between the corresponding large and small amplitude positive Wigner function peaks [75]. And in a classically chaotic regime, an initial coherent state will spread out in phase space, exhibiting a complex interference pattern of positive and sub- $\hbar$ (i.e., sub-Planckian) scale negative Wigner function regions [62]. However, depending on the environment temperature, such non-classical features will typically diffuse away for the usual device system-environment couplings, leaving a long time steady state that is closely approximated by the corresponding classical system Fokker-Planck equation.

Nevertheless, the question is still largely unresolved as to whether it might be possible to stabilize quantum states of a single mode bosonic system largely through its anharmonicities alone. In particular, for certain anharmonicity types and drives (whether externally or internally generated by the system dynamics), we may be able to prepare and maintain quantum states with significant associated negative Wigner function regions, despite the counteracting decoherence effects of environmental noise.

Of particular interest are nonlinear single mode systems such as the paradigmatic, driven damped Duffing oscillator. A number of investigations have employed the Wigner function representation to explore the resulting quantum phase space dynamics in parameter regimes where the corresponding classical nonlinear dynamics exhibits, for example, bistability or chaos [62, 61, 38, 57, 75]. By varying the system damping and noise (diffusion) due to coupling to the environment, the quantum to classical transition can be explored in a controllable and geometrically direct way by comparing the corresponding quantum Wigner function phase space and classical phase space pictures.

In the present chapter we will extend the Wigner formulation of open, single mode bosonic system quantum dynamics and take into account also the so-called Wigner current vector field (or 'Wigner current' in short) on phase space [11, 128, 17]. We will show that the Wigner current allows a particularly concise reformulation of the quantum Fokker-Planck equation as a standard continuity equation, equating (via the familiar Gauss's theorem of vector calculus) the rate of change of the net Wigner quasiprobability within some two dimensional region of phase space to the net Wigner current normal to the boundary enclosing the region.

The potential advantage of bringing the Wigner current into play is that it can give a graphic geometric representation of how non-classical states form through the system Hamiltonian anharmonicity, as well as diffuse away due to the environment.

By exploring the relative contributions to the net Wigner current across the boundary of a given negative region that arise from the system Hamiltonian anharmonicity and from the interactions with the environment, we may be able to improve our understanding of how to 'engineer' system Hamiltonian anharmonicities and drive tones so as to stabilize macroscopic bosonic quantum states in the presence of environmental noise. As an application of the geometric Wigner current construct for open quantum systems, the present chapter gives some initial steps in this direction.

Note that in the present chapter we do not attempt to address the largely open question as to how the negative Wigner function regions form in the first place; rather, we suppose that negative regions have already formed, and consider how the regions may be stabilized in the presence of environmental noise. Some promising first steps towards understanding how negative regions form from a Wigner current perspective are given in Refs. [104, 103].

In what follows, we reformulate the quantum Fokker-Planck equation for a one dimensional anharmonic particle system interacting with a thermal bath as a continuity equation in terms of the Wigner function and associated current. We then consider as specific system examples the harmonic oscillator and additively driven Duffing oscillator, solving numerically for their Wigner functions and currents. Some first steps are finally made towards a geometric understanding of how nonclassical states may be stabilized from a Wigner current perspective.
$\Gamma$ Section 3.1

## Moyal Bracket

We have already built up the necessary machinery to derive the phase-space quantum dynamical equation. Application of the Weyl transform to the von Neumann equation
takes us there. Recall that the von Neumann equation is

$$
\begin{equation*}
\frac{\partial \hat{\rho}}{\partial t}=-\frac{i}{\hbar}[\hat{H}, \hat{\rho}] . \tag{3.1}
\end{equation*}
$$

We have already studied how products of operators in Hilbert space transform into *-products in phase space. From Eq. (2.43) we have the defining equation for the Moyal bracket [97]:

$$
\begin{equation*}
\frac{\partial W(x, p)}{\partial t}=-\frac{i}{\hbar}\left[W_{\hat{H}} \star W(x, p)-W \star W_{\hat{H}}(x, p)\right]=\left\{W_{\hat{H}}, W\right\}_{M B} \tag{3.2}
\end{equation*}
$$

(Note that for this chapter only, we denote position with the variable $x$ instead of $q$.) For any two phase space functions $f$ and $g$, the Moyal bracket can also be written as

$$
\begin{equation*}
\{f, g\}_{M B}=\frac{2}{\hbar} f(x, p) \sin \left(\frac{\hbar}{2}\left(\overleftarrow{\partial}_{x} \vec{\partial}_{p}-\overleftarrow{\partial}_{p} \vec{\partial}_{x}\right)\right) g(x, p) \tag{3.3}
\end{equation*}
$$

This is the reason why the Moyal bracket is sometimes referred to as the 'Sine' bracket.
We will keep our discussion quite general with a Hamiltonian of the form $\hat{H}=$ $\hat{p}^{2} / 2 m+V(\hat{x})$. With the help of the Bopp shifts, this can be put in differential form as

$$
\begin{equation*}
\frac{\partial W}{\partial t}=-\frac{p}{m} \frac{\partial W}{\partial x}+\frac{\partial V}{\partial x} \frac{\partial W}{\partial p}+\sum_{n \geq 1} \frac{(-1)^{n}(\hbar / 2)^{2 n}}{(2 n+1)!} \frac{\partial^{2 n+1}}{\partial x^{2 n+1}} V \frac{\partial^{2 n+1}}{\partial p^{2 n+1}} W \tag{3.4}
\end{equation*}
$$

Notice how to the leading zeroth order in $\hbar$, we are left with just the Liouville equation of classical statistical mechanics with the Poisson bracket,

$$
\begin{equation*}
\frac{\partial W}{\partial t}=\frac{\partial H}{\partial x} \frac{\partial W}{\partial p}-\frac{\partial H}{\partial p} \frac{\partial W}{\partial x}=\{H, W\}_{P B} \tag{3.5}
\end{equation*}
$$

where $H=W_{\hat{H}}$ is the Weyl symbol of $\hat{H}$. As one might expect, the way to arrive at
classical dynamics in phase space is to formally set $\hbar=0$. Any term with a factor of $\hbar$ in it can be thought of as a quantum mechanical correction to the classical dynamics.

Importantly, not all quantum systems feature quantum corrections to the Moyal equation. Turning back to Eq. (3.4), the quantum harmonic oscillator with its potential that contains only a term of second-order in position will lead to the same equation whether inserted in the Moyal bracket or the Poisson bracket. In this scenario, every point in phase space will move according to the classical equations of motion. Here we will only be able to detect quantum mechanical properties by studying the state of the system at a single point in time. For harmonic potentials, quantum dynamical effects are absent.

## Section 3.2

## Wigner Current

The Moyal equation can be concisely rewritten as the continuity equation [11, 97, $128,142]$

$$
\begin{equation*}
\frac{\partial W}{\partial t}+\nabla \cdot \vec{J}=0 \tag{3.6}
\end{equation*}
$$

where $\nabla=\left(\partial_{x}, \partial_{p}\right)$ and the Wigner current $\vec{J}$ is implicitly defined as [11, 128]

$$
\begin{equation*}
\vec{J}=\binom{\frac{p}{m} W}{-\sum_{n=0}^{\infty} \frac{(-1)^{n}(\hbar / 2)^{2 n}}{(2 n+1)!} \partial_{x}^{(2 n+1)} V \partial_{p}^{(2 n)} W} . \tag{3.7}
\end{equation*}
$$

This recapitulation of the dynamical equation puts the focus on the flow of quasiprobability densities.

We can make a direct comparison to classical flows in phase space and we will now show that while classical dynamics in phase space are represented by flows that follow trajectories and behave as incompressible fluids, quantum flows may be compressible.

As one version of Liouville's theorem states, all classical particles and probability densities flow along trajectories determined by the Poisson bracket. Writing the Poisson bracket as a continuity equation, we get a classical current

$$
\begin{equation*}
\vec{J}_{\text {Classical }}=\binom{\frac{p}{m} P}{\partial_{x} V \cdot P} \tag{3.8}
\end{equation*}
$$

where $P$ is some classical probability distribution function. This classical current $\vec{J}_{\text {Classical }}$ can be decomposed into a product of a classical velocity vector field $\vec{v}$ and the probability distribution $P$ so we have

$$
\begin{equation*}
\vec{v}=\binom{\frac{p}{m}}{\partial_{x} V} \tag{3.9}
\end{equation*}
$$

The quantum phase space velocity field $\vec{w}$ can be defined similarly as [31]

$$
\begin{equation*}
\vec{w}=\vec{J} / W=\binom{\frac{p}{m}}{-\frac{1}{W} \sum_{n=0}^{\infty} \frac{(-1)^{n}(\hbar / 2)^{2 n}}{(2 n+1)!} \partial_{x}^{(2 n+1)} V \partial_{p}^{(2 n)} W} \tag{3.10}
\end{equation*}
$$

A key property when comparing the classical and quantum velocity vector fields is the divergence. The divergence of a velocity vector field captures the extent to which the density at some point in phase space is conserved as it flows along. Via the Divergence Theorem, one could also think of the density change as measured by the net flow velocity across the boundary of some infinitesimal test volume. A velocity vector field with zero divergence everywhere will transport some density (e.g. probability density or mass density of some liquid) such that it remains unchanged as it flows along the velocity field lines. Therefore, a fluid with a velocity vector field of zero divergence can also be called incompressible. Refering back to Eq. (3.9), we see
that $\nabla \cdot \vec{v}=0$ so classical systems behave as incompressible fluids in phase space. On the other hand, the quantum velocity field $\vec{w}$ may have a divergence that is not equal to zero because the momentum component in Eq. (3.10) may have a dependency on the momentum. This means that quantum flows in phase space may feature sources and sinks which give rise to nonlocal behavior for the quasiprobability.

As an example, consider some quantum system with a quartic potential in position $V(x)=\frac{1}{4} x^{4}$. The quantum velocity field for this case is

$$
\begin{equation*}
\vec{w}=\binom{\frac{p}{m}}{x^{3}+\frac{\hbar^{2}}{4} x \frac{\partial_{p}^{2} W}{W}} \tag{3.11}
\end{equation*}
$$

Furthermore, its divergence is nonzero:

$$
\begin{equation*}
\nabla \cdot \vec{w}=\frac{\hbar^{2}}{4} x \partial_{p}\left(\frac{\partial_{p}^{2} W}{W}\right) \neq 0 . \tag{3.12}
\end{equation*}
$$

As we emphasized previously, this holds true only for anharmonic systems. If the system has a potential that is a second order power of position at most, the flow is conservative and behaves as a incompressible fluid. In these cases, quasiprobability densities follow classical trajectories in phase space. This can be particularly useful in a numerical setting because the Wigner function can be propagated in time using massively parallel simulations of many different launched trajectories. However, it is crucial not to apply these same methods to the case of anharmonic systems. A serious issue arises in these cases where the necessary conditions for the existence of a unique solution to the differential equations at play is not satisfied [103].

We are left with a series of natural questions. How is the Wigner function normalization globally preserved? How do negative regions form? Where do negative regions form? The tomographic set of marginals determines the phase space function.

If we know only a subset of the marginal distributions and also where large (on the Planck scale) positive regions exist, can we then predict where significant negative regions will appear?

These are all difficult questions that we do not have answers to as of yet. However, they serve as strong motivation for studying representations of finite dimensional quantum systems in a discrete phase space where the appearance of negativity may be more directly tracked. This is the content of chapters 4-6.

## Section 3.3

## Quantum Fokker-Planck Equation

We now turn to open quantum systems where some noisy environment may interact with a quantum apparatus. For a one-dimensional, mass $m$ particle with Hamiltonian $H=p^{2} /(2 m)+V(x, t)$, where $V(x, t)$ is the (time dependent) potential energy, a suitable Lindblad master equation that describes the quantum dynamics of the system state characterized by density matrix $\rho(t)$ interacting with an oscillator bath can be written as follows:

$$
\begin{align*}
\frac{d \rho}{d t}= & -\frac{i}{\hbar}[H, \rho]+\frac{\gamma}{2}(\bar{n}+1)\left(2 a \rho a^{\dagger}-a^{\dagger} a \rho-\rho a^{\dagger} a\right) \\
& +\frac{\gamma}{2} \bar{n}\left(2 a^{\dagger} \rho a-a a^{\dagger} \rho-\rho a a^{\dagger}\right), \tag{3.13}
\end{align*}
$$

where $\gamma$ is the system energy damping rate and $\bar{n}=\left(e^{\hbar \omega_{0} /\left(k_{B} T\right)}-1\right)^{-1}$ is the BoseEinstein thermal average occupation number of the temperature $T$ bath at the characteristic harmonic oscillation frequency $\omega_{0}$ of the system Hamiltonian. Strictly speaking, the master equation (3.13) is valid to a good approximation provided the system-environment interaction is weak: $\gamma \ll \omega_{0}$, the temperature is in the range $\hbar \gamma \ll k_{B} T \ll \hbar \omega_{0}$, and the anharmonic potential contribution $V-m \omega_{0}^{2} x^{2} / 2$ is suf-
ficiently weak [60]. However, following frequent practice, we will assume that the master equation can still give reasonable open system quantum dynamics even when these conditions are not strictly adhered to.

Using the now familiar method of transforming Hilbert space operators into phase space functions, we can express the master equation (3.13) in terms of the Wigner function. We have already seen the phase space representation of the first term giving the closed system dynamics as the Moyal bracket. The interaction terms with the environment transform so that in total we have

$$
\begin{align*}
\frac{\partial W}{\partial t}= & -\frac{p}{m} \frac{\partial W}{\partial x}+\frac{\partial V}{\partial x} \frac{\partial W}{\partial p}  \tag{3.14}\\
& +\sum_{n \geq 1} \frac{(-1)^{n}(\hbar / 2)^{2 n}}{(2 n+1)!} \frac{\partial^{2 n+1}}{\partial x^{2 n+1}} V \frac{\partial^{2 n+1}}{\partial p^{2 n+1}} W  \tag{3.15}\\
& +\frac{\gamma}{2} \frac{\partial}{\partial x}\left[x W+\hbar\left(\bar{n}+\frac{1}{2}\right) \frac{1}{m \omega_{0}} \frac{\partial W}{\partial x}\right]  \tag{3.16}\\
& +\frac{\gamma}{2} \frac{\partial}{\partial p}\left[p W+\hbar\left(\bar{n}+\frac{1}{2}\right) m \omega_{0} \frac{\partial W}{\partial p}\right] \tag{3.17}
\end{align*}
$$

This so-called 'quantum Fokker-Planck' equation can still be written as a continuity equation. The Wigner current vector fields for the system and environment [17] are defined respectively as follows:

$$
\begin{equation*}
\mathbf{J}_{\mathrm{sys}}=\binom{\frac{p}{m} W}{-\sum_{n=0} \frac{(-1)^{n}(\hbar / 2)^{2 n}}{(2 n+1)!} \partial_{x}^{(2 n+1)} V \partial_{p}^{(2 n)} W} \tag{3.18}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{J}_{\mathrm{env}}=-\frac{\gamma}{2}\binom{x W+\hbar\left(\bar{n}+\frac{1}{2}\right)\left(m \omega_{0}\right)^{-1} \partial_{x} W}{p W+\hbar\left(\bar{n}+\frac{1}{2}\right) m \omega_{0} \partial_{p} W} \tag{3.19}
\end{equation*}
$$

where the first row is the position $x$ component and the second row is momentum $p$
component of the current vector. The environment current can be further decomposed as a sum of damping and diffusion contributions: $\mathbf{J}_{\text {env }}=\mathbf{J}_{\text {damp }}+\mathbf{J}_{\text {diff }}$, where

$$
\begin{equation*}
\mathbf{J}_{\mathrm{damp}}=-\frac{\gamma}{2}\binom{x W}{p W} \tag{3.20}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{J}_{\mathrm{diff}}=-\frac{\gamma \hbar}{2}\left(\bar{n}+\frac{1}{2}\right)\binom{\left(m \omega_{0}\right)^{-1} \partial_{x} W}{m \omega_{0} \partial_{p} W} \tag{3.21}
\end{equation*}
$$

In terms of the system and environment currents, the master equation for the Wigner function (3.17) takes the concise form of a continuity equation:

$$
\begin{equation*}
\frac{\partial W}{\partial t}+\nabla \cdot \mathbf{J}=0 \tag{3.22}
\end{equation*}
$$

where $\mathbf{J}=\mathbf{J}_{\text {sys }}+\mathbf{J}_{\text {env }}$ and $\nabla=\left(\partial_{x}, \partial_{p}\right)$.

## - Section 3.4

## Harmonic and Duffing Oscillator Wigner

## Currents

The driven Duffing oscillator is characterized by the anharmonic + additive driving potential

$$
\begin{equation*}
V(x, t)=\frac{1}{2} m \omega_{0}^{2} x^{2}+\frac{\lambda}{4} x^{4}-x F \cos \left(\omega_{d} t\right) \tag{3.23}
\end{equation*}
$$

where the parameter $\lambda$ gives the strength of the anharmonic potential, the parameter $F$ gives the strength of the time-dependent sinusoidal drive, and $\omega_{d}$ is the drive frequency. Substituting Eq. (3.23) into Eq. (3.18), we obtain for the driven Duffing
oscillator system Wigner current:

$$
\begin{equation*}
\mathbf{J}_{\text {Duff }}=\binom{\frac{p}{m} W}{\left[-m \omega_{0}^{2} x+F \cos \left(\omega_{d} t\right)-\lambda x^{3}+\frac{\hbar^{2} \lambda}{4} x \partial_{p}^{2}\right] W} . \tag{3.24}
\end{equation*}
$$

For the harmonic oscillator the system Wigner current simplifies to

$$
\begin{equation*}
\mathbf{J}_{\mathrm{HO}}=\binom{\frac{p}{m} W}{-m \omega_{0}^{2} x W} \tag{3.25}
\end{equation*}
$$

It is convenient to work in terms of dimensionless forms of the Wigner function and current. In terms of the length unit $x_{0}=\sqrt{\hbar /\left(m \omega_{0}\right)}$ and time unit $t_{0}=\omega_{0}^{-1}$, we transform the various coordinates and parameters into dimensionless form as follows: $\tilde{x}=x / x_{0}, \tilde{p}=p /\left(m \omega_{0} x_{0}\right), \tilde{F}=x_{0} F /\left(\hbar \omega_{0}\right), \tilde{\lambda}=\lambda x_{0}^{4} /\left(\hbar \omega_{0}\right), \tilde{\gamma}=\gamma / \omega_{0}, \tilde{\omega}_{d}=\omega_{d} / \omega_{0}$, and $\tilde{t}=\omega_{0} t$, where the tilde denotes the dimensionless form. The dimensionless form for the Wigner function is

$$
\begin{align*}
\tilde{W} & =\hbar W \\
& =\frac{1}{\pi} \int_{-\infty}^{+\infty} d y e^{-2 i p y / \hbar}\langle x+y| \rho(t)|x-y\rangle \\
& =\frac{1}{\pi} \int_{-\infty}^{+\infty} d \tilde{y} e^{-2 i \tilde{p} \tilde{y}}\langle\tilde{x}+\tilde{y}| \rho(t)|\tilde{x}-\tilde{y}\rangle, \tag{3.26}
\end{align*}
$$

where $|\tilde{x}\rangle=\sqrt{x_{0}}|x\rangle$ [so that $\left.\left\langle\tilde{x} \mid \tilde{x}^{\prime}\right\rangle=\delta\left(\tilde{x}-\tilde{x}^{\prime}\right)\right]$. The continuity equation becomes in dimensionless form:

$$
\begin{equation*}
\frac{\partial \tilde{W}}{\partial \tilde{t}}+\tilde{\nabla} \cdot \tilde{\mathbf{J}}=0 \tag{3.27}
\end{equation*}
$$

where $\tilde{\mathbf{J}}=\tilde{\mathbf{J}}_{\text {Duff }}+\tilde{\mathbf{J}}_{\text {env }}$, with

$$
\begin{equation*}
\tilde{\mathbf{J}}_{\mathrm{Duff}}=\binom{\tilde{p} \tilde{W}}{\left[-\tilde{x}+\tilde{F} \cos \left(\tilde{\omega}_{d} \tilde{t}\right)-\tilde{\lambda} \tilde{x}^{3}+\frac{\tilde{\lambda}}{4} \tilde{x} \partial_{\tilde{p}}^{2}\right] \tilde{W}} \tag{3.28}
\end{equation*}
$$

and

$$
\begin{equation*}
\tilde{\mathbf{J}}_{\text {env }}=\tilde{\mathbf{J}}_{\mathrm{damp}}+\tilde{\mathbf{J}}_{\text {diff }}, \tag{3.29}
\end{equation*}
$$

with

$$
\begin{equation*}
\tilde{\mathbf{J}}_{\text {damp }}=-\frac{\tilde{\gamma}}{2}\binom{\tilde{x} \tilde{W}}{\tilde{p} \tilde{W}} \tag{3.30}
\end{equation*}
$$

and

$$
\begin{equation*}
\tilde{\mathbf{J}}_{\mathrm{diff}}=-\frac{\tilde{\gamma}}{2}\left(\bar{n}+\frac{1}{2}\right)\binom{\partial_{\tilde{x}} \tilde{W}}{\partial_{\tilde{p}} \tilde{W}} \tag{3.31}
\end{equation*}
$$

For the harmonic oscillator, we have for the dimensionless current: $\tilde{\mathbf{J}}=\tilde{\mathbf{J}}_{\mathrm{HO}}+\tilde{\mathbf{J}}_{\text {env }}$, with

$$
\begin{equation*}
\tilde{\mathbf{J}}_{\mathrm{HO}}=\binom{\tilde{p} \tilde{W}}{-\tilde{x} \tilde{W}} \tag{3.32}
\end{equation*}
$$

and $\tilde{\mathbf{J}}_{\text {env }}$ given by Eq. (3.29) From now on, we drop the tildes for notational convenience, the dimensionless form of the parameters and coordinates understood.

In Figs. 3.1-3.4, we show example numerical solutions to the Wigner function $W$ and associated current vector field $\mathbf{J}$ for the undriven, open harmonic and driven Duffing oscillator systems. This involves first solving the Lindblad master equation (3.13)
for the system density matrix $\rho(t)$ using QuTiP [72] and then evaluating the Wigner function and current in terms of the density matrix. Although the Wigner function time dependence for the open harmonic oscillator system can be determined analytically [77, 106], we nevertheless solve the harmonic oscillator master equation numerically as a check on the validity of our code.

Figure 3.1 shows snapshots of the evolving Wigner function and associated current $\mathbf{J}=\mathbf{J}_{\mathrm{HO}}+\mathbf{J}_{\text {env }}$ for the harmonic oscillator initially in an initial superposition of coherent states separated by $x=6$; the snapshot times are given in multiples of the free oscillation period $\tau=2 \pi / \omega_{0}=2 \pi t_{0}$. The damping rate is chosen to be $\gamma=0.01$ and the bath temperature is set to zero. Regions color-coded blue correspond to positive Wigner function value, red regions correspond to negative Wigner function value, while the local color density gives a measure of the Wigner function magnitude. A unit area square corresponding to Planck's constant $\hbar$ in our dimensionless units is indicated at the bottom right of each figure to give the scale, while the arrow legend at the top left of each figure indicates the scale for the current vector field. Figure 3.2 shows the same evolving Wigner function snapshots as in Fig. 3.1 but with just the environmental diffusion current $\mathbf{J}_{\text {diff }}$ (3.31) indicated. In the final indicated snapshots corresponding to $t=100 \tau$ [Figs. 3.1-3.2(c)], the Wigner function and current hardly change between subsequent snapshots separated by a free oscillation period, indicating that the system dynamics has reached a steady state to a good approximation. This is to be expected given that $\gamma t=2 \pi$, i.e., the final snapshot time is approximately six times longer than the harmonic oscillator relaxation time.

Figure 3.3 shows snapshots of the evolving Wigner function and associated current $\mathbf{J}=\mathbf{J}_{\text {Duff }}+\mathbf{J}_{\text {env }}$ for the driven Duffing oscillator initially in an undisplaced coherent state; the snapshot times are given in multiples of the drive period $\tau_{d}=2 \pi / \omega_{d}$. We choose the dimensionless Duffing oscillator parameter values $\lambda=0.05$ (anharmonic


Figure 3.1: Snapshots of evolving harmonic oscillator Wigner function and associated current vector field $\mathbf{J}=\mathbf{J}_{\mathrm{HO}}+\mathbf{J}_{\text {env }}$ for an initial superposition of coherent states with separation $x=6$; the damping rate $\gamma=0.01$ and bath temperature $T=0$.


Figure 3.2: Snapshots of evolving harmonic oscillator Wigner function and associated environmental diffusion current vector field $\mathbf{J}_{\text {diff }}$ for an initial superposition of coherent states with separation $x=6$; the damping rate $\gamma=0.01$ and bath temperature $T=0$.
strength), $\omega_{d}=1.09$ (drive frequency), $F=0.092$ (drive strength), and $\gamma=0.01$ (damping rate), with the bath temperature set to zero. These parameter values result in the classical Duffing oscillator exhibiting bistability for the steady state dynamics at zero temperature, corresponding to coexisting small and large amplitude oscillations. For the above parameter choices, these small and large steady state amplitudes are 0.52 and 2.46 , respectively. Figure 3.4 shows the same evolving Wigner function snapshots as in Fig. 3.3, but with just the environmental diffusion current $\mathbf{J}_{\text {diff }}$ (3.31) indicated. In the final indicated snapshots corresponding to $t=300 \tau_{d}$ [Figs. 3.3$3.4(\mathrm{c})$ ], the Wigner function and current hardly change between subsequent snapshots separated by a drive period. These final snapshots should therefore correspond fairly accurately to the long time limit steady state Wigner function and current.


Figure 3.3: Snapshots of evolving Duffing oscillator Wigner function and associated current vector field $\mathbf{J}=\mathbf{J}_{\text {Duff }}+\mathbf{J}_{\text {env }}$ for an initial undisplaced coherent state; the damping rate $\gamma=0.01$ and bath temperature $T=0$.

## Section 3.5

## Geometric Investigation

Common to the harmonic and Duffing oscillator quantum dynamics indicated in Figs. 3.1 and 3.3 , the direction of the current $\mathbf{J}$ in the regions of positive-valued Wigner function is clockwise about the phase space origin, just as is the case for


Figure 3.4: Snapshots of evolving Duffing oscillator Wigner function and associated environmental diffusion current vector field $\mathbf{J}_{\text {diff }}$ for an initial undisplaced coherent state; the damping rate $\gamma=0.01$ and bath temperature $T=0$.
an evolving classical probability density that results from solving the corresponding classical Fokker-Planck equation for some initial probability distribution; for the harmonic oscillator system, the Wigner current continuity equation (3.22) coincides with the classical, Brownian motion Fokker-Planck equation, while for the Duffing oscillator the Wigner current continuity equation (3.22) differs from the classical Fokker-Planck equation only in the presence of the system quantum current term $\left(0, \lambda x \partial_{p}^{2} W / 4\right)$ [see Eq. (3.28)]. In contrast, the current direction in the regions of negative-valued Wigner function is counterclockwise, i.e., in the opposite direction to the corresponding classical current $[11,128,2]$.

The harmonic and time-dependent drive contributions to $\mathbf{J}_{\text {Duff }}[\mathrm{Eq}$. (3.28)] dominate the Wigner current. In principle, one can go to the rotating frame of the drive so that the nontrivial anharmonic and environment contributions dominate. However, the resulting algebraic expressions in the rotating frame become quite involved. While considerable simplification results if the rotating wave approximation (RWA) is made, we prefer not to go to the rotating frame and perform the RWA in the present work. As we shall see below, the dynamics throughout a given drive period of the non-vanishing quantum and diffusion contributions to the Wigner current on a negative region boundary suggests a possible way to stabilize negative regions in the
long time limit; such a dynamics would be effectively averaged over if the RWA were carried out.

In Figs. 3.2 and 3.4, we can see that for any negative-valued Wigner function region, the diffusion contribution to the environmental current $\mathbf{J}_{\text {diff }}$ is always directed inwards on the boundary of the negative region, with the result that the environmental diffusion current acts to destroy negative regions. This is just the process of decoherence viewed in terms of the Wigner current.

In order to gain a better understanding of the Wigner function evolution for nonclassical states, let us suppose that the Wigner function at some given time instant $t$ is negative in certain regions of phase space. This is the case for the initial coherent state superposition example considered above (see Figs. 3.1 and 3.2), while for the Duffing oscillator, we see that negative Wigner function regions are generated through the dynamics (Figs. 3.3 and 3.4). Consider a particular negative region with phase space area $A(t)$ and boundary $\partial A(t)$, where the indicated $t$-dependence accounts for the fact that the negative region evolves in time. In particular, the boundary is defined by $\left.W(x, p, t)\right|_{\partial A(t)}=0$. A measure of the degree of negativity of the region is given by the negative 'volume' under the integral $\int_{A(t)} d x d p W(x, p, t)$. From Eqs. (3.27)-(3.31) and Gauss's theorem, the time rate of change of this negative volume is

$$
\begin{align*}
& \frac{d}{d t} \int_{A(t)} d x d p W(x, p, t)=-\frac{\lambda}{4} \int_{\partial A(t)} d s \mathbf{n} \cdot(0, x) \frac{\partial^{2} W}{\partial p^{2}} \\
& +\frac{\gamma}{2}\left(n+\frac{1}{2}\right) \int_{\partial A(t)} d s \mathbf{n} \cdot \nabla W \tag{3.33}
\end{align*}
$$

where we have used the fact that the Wigner function vanishes on the boundary $\partial A(t), s$ parametrizes the boundary curve, and $\mathbf{n}$ is the unit vector outwards normal to the curve.

For the harmonic oscillator system, the first term on the right hand side of the
equals sign in Eq. (3.33) vanishes (since $\lambda=0$ ) and the rate of change of the region negativity is affected solely by the environmental diffusion current (3.31). Since the Wigner function is by definition negative on the interior region and positive on at least the immediate exterior region of the boundary $\partial A(t)$, the gradient $\nabla W$ points outwards so that $\mathbf{n} \cdot \nabla W \geq 0$ everywhere on the boundary. Therefore, for the harmonic oscillator we have that

$$
\begin{equation*}
\frac{d}{d t} \int_{A(t)} d x d p W(x, p, t) \geq 0 \tag{3.34}
\end{equation*}
$$

and we thus see that the volume of a negative region always decreases with time at a rate governed by the environmental diffusion current. That the environment causes decoherence for a harmonic oscillator initially in a quantum superposition state is of course well-known. Nevertheless, in our view there is value in picturing the process of decoherence from a geometric, current perspective, especially in the case where the oscillator is anharmonic.

In particular, for the Duffing oscillator system $(\lambda \neq 0)$ on the other hand, we see from Eq. (3.33) that the rate of change of the region negativity is now governed by two current contributions: the system quantum current $\left(0, \lambda x \partial_{p}^{2} W / 4\right)$ and the environmental diffusion current (3.31). Figure 3.5 shows the current flow vector field on the $W=0$ boundary of a particular negative region, comprising the net sum of these two current contributions. Each row of subplots in Fig. 3.5 show the Wigner function at times separated by one quarter of the drive period. The subplots in the first row occur at times when the long-term trend is for the negativity volume to grow. Those in the second and third rows occur at times with the opposite long-term behavior. Figure $3.5(\mathrm{a})$ shows a negative region when its volume is slightly increasing due to a net outward boundary current. The region evolves into that which is shown in Fig. 3.5(b) when the negativity volume is now significantly increasing due to the comparatively larger net outward boundary current. In Fig. 3.5(c) the region now


Figure 3.5: Snapshots of evolving Duffing oscillator Wigner function and current vector field $\mathbf{J}$ on the boundary of a single negative region for an initial undisplaced coherent state; the damping rate $\gamma=0.01$ and bath temperature $T=0$.


Figure 3.6: Time evolution for the volume of the negative region which is the focus of Fig. 3.5. The arrows along the horizontal axis indicate approximate times corresponding to the snapshots appearing in the Fig. 3.5 rows.
contracts with the net flow inwards but expands again in Fig. 3.5(d). This pulsating behavior repeats itself over many periods of the drive, as can be seen in the other plots of Fig. 3.5 and also in Fig. 3.6 which traces the continuous evolution of the same negative region from initial formation to almost complete disappearance; note in particular the small half-drive period oscillations in the negativity volume.

At a given location on the boundary, the environmental diffusion current always flows inwards, thus acting to destroy the negative region. On the other hand, the functional dependence of the system quantum current leads to more varied flow behavior on the boundary and can act to either create or destroy the negative regions; the quantum current must therefore be responsible for the initial generation and possible eventual stabilization of negative regions in the steady state.

In order to counteract the diffusive inflow, from Eq. (3.33) we necessarily require that

$$
\begin{equation*}
\frac{\lambda}{4} \int_{\partial A(t)} d s \mathbf{n} \cdot(0, x) \frac{\partial^{2} W}{\partial p^{2}}>0 \tag{3.35}
\end{equation*}
$$

Although the orientation and location of the boundary segments along with the magnitude of the term $\partial_{p}^{2} W$ can result in quite complicated current flows on the negative region boundaries, there are certain conditions that lead to predictable flow patterns. To aid our intuition, we will focus on the two terms $\mathbf{n} \cdot(0, x)$ and $\partial_{p}^{2} W$ in Eq. (3.35), which must both be large for there to be a significant quantum current. The $\mathbf{n} \cdot(0, x)$ term is significant on boundary segments with sizable $x$ coordinates and where the normal vector $\mathbf{n}$ is oriented with a small angle relative to the vertical $p$ axis. At certain times in the evolution over a given drive period, the second derivative $\partial_{p}^{2} W$ becomes large at these same boundary segments, resulting in sizable boundary currents inwards or outwards depending on the relative signs of the $\mathbf{n} \cdot(0, x)$ and $\partial_{p}^{2} W$ terms, and the sign of $\lambda$ which we assume here to be positive. This can be observed whenever the large positive peak is directly above or below the negative region of
interest [c.f. Figs. $3.5(\mathrm{~b})$, (d), (f), (h), (j), and (l)], leading to large positive $\partial_{p}^{2} W$ where $\mathbf{n} \cdot(0, x)>0$, and therefore resulting in a large net outward current on the boundary segment of the negative region that is proximal to the large positive peak.

In the quadrants where $x$ and $p$ have the same sign [c.f. Figs. 3.5(a), (c), (e), $(\mathrm{g})$, ( i ), and (k)], the positive peak is now directly to the left or right of the negative region of interest, leading instead to large positive $\partial_{x}^{2} W$, and having little effect on the $\partial_{p}^{2} W$ term. In fact, in these quadrants, $\partial_{p}^{2} W<0$ on most of the negative region boundary where $\mathbf{n} \cdot(0, x)>0$, resulting in a net inwards flow.

This change in the direction of the quantum current from outwards to inwards on the boundary is the mechanism responsible for the oscillation in the volume of the individual negative region as it cycles clockwise in phase space. Figure 3.6 clearly displays this "heartbeat" behavior for the single negative region of interest shown in the snapshots of Fig. 3.5. Arrows on the time axis of Fig. 3.6 indicate the approximate times of the snapshots.

While significant negative regions develop in the Duffing oscillator numerical example considered above during intermediate times [Fig. 3.3(b)], the negative regions practically vanish in the long time limit steady state, even at zero temperature [Fig. 3.3(c)]. We have seen that as negative regions transit between adjacent quadrants of phase space, there is an ebb and flow in their size; any stabilization of negativity would be in the sense that significant negativity persists and is repeated at times that are equal modulo the drive period. For this to be possible, the net effect of the quantum current over the drive period must be to increase the negativity volume, thus counteracting the deleterious effects of the diffusion term in Eq. (3.33). If the negative region does not disappear during one period of the drive $\tau_{d}$, we can


Figure 3.7: Snapshots of an idealized Wigner distribution function geometry that may sustain a significant negative region as it repeatedly cycles clockwise in phase space. Note that the indicated distribution is schematic only and meant to show the relative locations of the positive and negative regions and their overall aspect ratios. The distribution is squeezed in such a way as to maximize the horizontal dimension when the quantum current can play a dominant role in growing the negative volume, and minimize the horizontal dimension when the net Wigner current would otherwise reduce the negative volume.
formulate a criterion for this weaker form of stabilization as follows:

$$
\begin{equation*}
\int_{t_{0}}^{t_{0}+\tau_{d}} d t\left[-\frac{\lambda}{4} \int_{\partial A(t)} d s \mathbf{n} \cdot(0, x) \frac{\partial^{2} W}{\partial p^{2}}+\frac{\gamma}{2}\left(n+\frac{1}{2}\right) \int_{\partial A(t)} d s \mathbf{n} \cdot \nabla W\right] \leq 0 \tag{3.36}
\end{equation*}
$$

as $t_{0} \rightarrow \infty$.
In the above simulations the chosen, example parameter values result in coexisting small and large amplitude stable oscillations for the classical dynamics; the Wigner function must correspondingly spread out through diffusive current flow from its initially narrow and strongly peaked coherent state distribution [Fig. 3.3(a)]. As a result, the magnitude of the term $\partial_{p}^{2} W$ must decrease overall, and with the small chosen anharmonic coupling strength value $\lambda(=0.05)$, the system quantum current term is too weak to counter the deleterious effects of the diffusion term in Eq. (3.33) and hence be able to stabilize sizable negative regions.

Although we have not yet identified specific examples that lead to the evolution of the Wigner function such that Eq. (3.36) is satisfied, there are certain, basic Wigner distribution function geometries that appear to be good candidates. In particular, consider the situation where a negative region precedes a large positive region as they cycle clockwise in phase space, similar to what is observed in actual simulations as we have discussed above. If, furthermore, the positive region is squeezed such that a large portion of its boundary is proximal to the negative region [c.f. Fig. 3.7], then according to our above analysis there should be significant growth of the negative region volume in the two quadrants where $x$ and $p$ have opposite sign, while in the other two quadrants where $x$ and $p$ have the same sign there should be comparatively less shrinkage of the negative region volume. Note that the preceding argument depends on the inflection point $\partial_{p}^{2} W=0$ occurring where $W>0$, so that $\partial_{p}^{2} W>0$ on the negative region boundary; based on our above numerical simulations of the Duffing oscillator, it is reasonable to assume this over a large portion of the boundary
provided that the maximum of the positive peak is relatively large compared to the minimum of the adjacent negative region.

What other shapes might optimize this orientation of negative and positive regions? We can at least think of designing transient states with negative regions that might be quite long lived. One example is of the shape of a narrow propeller of negativity as seen in Fig. 3.8. As it spins in phase space, it may be quite long lived due to the fact that it has a maximal horizontal boundary at the optimal time when the positive region is above it. This is an orientation that maximizes the second order derivative with respect to momentum, which is a key factor in the quantum current.


Figure 3.8: Idealized 'propeller' shape of negative region that would rotate clockwise in phase space. This geometry maximizes the edge length perpendicular to momentum direction when the positive peak is advantageously positioned directly above or below. Negative region loss may be minimized when positive peaks are offset from the negative region in the position direction because the edge length perpendicular to the momentum direction is minimized.

This mode of thinking is appealing because it gives us some geometric intuition which is usually hard to develop when thinking about quantum problems. Furthermore, one could explore the quantum current terms from a number of different higher order nonlinearities. With this approach, different orders of derivatives of the Wigner function with respect to momentum would become important.

(c)


Figure 3.9: Cross-section in the momentum direction at a positive position coordinate of a positive and negative peak. The sign of the curvature in the momentum direction can be seen to correlate with the relative heights of these peaks. We can see that the curvature at the edge of the negative region closest to the positive peak is greatest for (a), decreases to zero for (b), and becomes negative for (c).

A major concern in the driven Duffing setting is the diffusion of just the positive quasiprobability. This can be related to the initial rise and then fall of negativity we see. Initially we have a strong coherent state. Negativity immediately starts to form at the leading edge of the positive peak as it rotates around the origin. Formation of negativity means there must be formation of extra positive quasiprobability due
to normalization. However, the extra positive quasiprobability does not form only in the positive peak, which would maintain the second order derivative with respect to momentum at the interface with the negative region between the peaks. It must wrap around the sides of the negative region to some extent in order to keep the marginals non-negative. This means that the ratio of the height magnitudes corresponding to the positive and negative peaks gets smaller and smaller as the negativity grows. The curvature in the momentum direction inevitably decreases, reducing the system current which is necessary for negativity to be sustained. Figure 3.9 shows a crosssection cut in the momentum direction which displays how this curvature becomes disadvantageous as the negative region becomes large relative to the positive region. It shows how reducing the relative magnitudes of the peak heights leads to a reduction in the curvature necessary to keep current flowing out of the negative region.

The diffusion current is problematic at the negative region boundary, but we can now see how it can be a problem elsewhere as well. The diffusion current also causes the positive peaks to reduce in size and positive quasiprobability flows towards the classical fixed points. As we have already seen, any decrease in the positive peak will have a first order effect of reducing the magnitude of the momentum curvature, thus throttling the system current necessary to keep the negative region alive. Although no hard conclusions have been drawn here, we hope the above considerations may prompt some future geometric investigations relating to stabilization of macroscopic negativity or a no-go theorem preventing such stabilization.

- Section 3.6


## Summary

In this chapter, we extended the Wigner phase space formulation of open quantum system dynamics to include a description of the Wigner current vector fields on phase
space. This enables the quantum Fokker-Planck equation describing the Wigner function dynamics to be written in the concise form of a continuity equation. The evolving Wigner current was investigated numerically for a harmonic oscillator and a driven Duffing oscillator in the bistable regime, the latter serving as an illustrative anharmonic system. Through the application of the two-dimensional Gauss's theorem to boundary-enclosed, negative Wigner function regions on system phase space, we saw that the growth and reduction of negative regions are governed solely by the so-called quantum current due to the system anharmonicity and the diffusion current across the negative region boundaries. By examining the geometric form of these specific contributions to the total Wigner current, we were able to gain some initial insights as to how negative regions might be stabilized, i.e., maintained in the steady state. There exist many promising experimental systems where these considerations may lead to Wigner functions with relatively long-lived negative regions and border on the macroscopic regime. Examples include superconducting circuit microwave oscillators $[35,66,140,91,39,40,124,123,78,139]$ and opto- and electromechanical systems $[111,99,112,135,1]$. Levitated systems are particularly exciting with the recent advance of an optically trapped nanoparticle cooled close to its ground state while also exerting temporal control over the trapping potential [36].

## Chapter 4

## Discrete Phase Space

## - Section 4.1

## Introduction

The discussion thus far has considered quantum systems with continuous position coordinate space. This is certainly how much of the world around us appears. Most particles or objects move through a continuum of positional states. Differential calculus was indeed discovered to facilitate the mathematical description of this state of affairs.

However, it may be the case that we think of classical objects as belonging to a discrete state space such as a coin lying either heads of tail up. This can result from our inability to understand some intermediate states of a system in a meaningful way. Often we are left with a probabilistic description of the discrete space. As an example, consider a coin flipped up in the air above a table. Recording the first seconds of its trajectory with a modern high-speed camera, we can certainly evaluate its initial position and momentum of the center of mass as well as the angular momentum. We can predict the bounces it will make off the table and have a good idea of whether heads or tails will land up. However, a small air current will perturb the flight, our
table will not be perfectly smooth, we will slightly misjudge the inelasticity of the table, our idealization of the coin as rigid was incorrect and it flexed slightly upon impact with the table dissipating some energy as heat, and there will also be some error in our initial estimation of the state of the coin. The combination of these effects can scramble any expectation we might have about the final resting state. In the end, we may only be able to build up statistics from repeated trials in order to give a probabilistic prediction of the resting state of the coin.

Quantum mechanics introduces an analog of this idea, but one that is intrinsic to the quantum description. The prototypical example is the spin of an electron whose standard mathematical description is set in a two dimensional complex Hilbert space. Up until now, we have spent a great deal of effort motivating and explaining the benefits of describing quantum mechanics on phase space in order to construct tools to delineate the quantum and classical worlds. In the discrete setting, it is not as natural to think of the classical world as set on a phase space. Momentum arises from the rate of continuous displacement in time of a continuous position variable; we are now doing away with continuity.

As with the standard Wigner function, there exists a mapping from Hilbert space to a phase space that can be interpreted as a joint quasi-probability distribution over two Fourier conjugate variables. The major difference is that instead of a continuous Hilbert space, we start with a description of our system in a finite $N$-dimensional Hilbert space, where $N$ is some integer. There are in fact many ways of accomplishing such a transformation resulting in phase spaces with different properties. This chapter reviews what is known about one method of constructing a discrete Wigner function whose dynamics we explore in the following two chapters.

Although he was not directly concerned with discrete Wigner functions, Schwinger developed a complete orthonormal basis of $N^{2}$ unitary operators [119, 120, 121, 122].

These form a representation of the Heisenberg-Weyl group modulo its center, a fact that we will explore in depth. Once we have an orthonormal operator basis, we may describe the state of our system using the coefficients found in an expansion of the density operator with respect to our basis. Choosing the best such orthonormal basis that leads to the required mathematical properties of a Wigner function is the task laid out in this chapter. Following this line of thinking, Buot [20] introduced a Weyl transformation that generates a Wigner function on the toroidal $N \times N$ lattice (with $N$ odd). Hannay and Berry [64] adopted a different method whereby the familiar continuous Wigner function is manipulated to define a discrete Wigner function on a $2 N \times 2 N$ lattice.

These two basic approaches were further developed and rediscovered by others. Cohen and Scully [26] and Feynman [44] investigated the case of $N=2$. Wootters [143], Galetti and De Toledo Piza [48], and Cohendet et al. [27] made subsequent investigations of the $N \times N$ case. The $2 N \times 2 N$ model saw further attention from Leonhardt [83, 84, 85]. Gibbons et al. [53] extended previous formulations of discrete Wigner functions that work naturally for $N$ equal to prime dimensional systems to the case of prime powers using finite fields. Extension to the case of all odd dimensions was performed by Gross [59]. Ferrie and Emerson [43] used the theory of mathematical frames to generalize the process of finding a Wigner function and defined the set of possible discrete Wigner functions.

There are numerous discrete Wigner function formulations and applications of which we only mention a representative few here. Applications of Wigner functions in discrete phase space include quantum optics [133], algorithms for quantum computation [13, 94], analysis of quantum computational speedup [50, 29], quantum error correcting codes [107], decoherence in quantum walks [90], quantum teleportation [105, 93], and investigations of quantum Bayesianism [22, 47, 46]. There exist a num-
ber of good reviews containing many more references [74, 113, 136, 42]. We follow Vourdas [138] in sections 4.4-4.8.

Our results presented in the two chapters that follow this one provide a foundation for how to understand the dynamics of quantum systems in discrete phase spaces. Although we have noted that there are generalizations to all dimensions of Hilbert space, these introduce unnecessary complications for initial investigations. There is much we can learn just about the dynamics using the most straight-forward and intuitive formulation of the discrete Wigner function that applies to prime dimensional quantum systems due to Wootters [143]. The major benefit of this approach is the duality between the physical description of prime dimensional quantum systems and the geometric properties of the corresponding discrete phase space.

The physical motivation may appear lacking but we can appeal to a mathematical one for now. The continuous Fourier transform is the mapping that connects continuous position and momentum. It is also the mathematical origin of the Heisenberg uncertainty principle which many consider a signature of quantum behavior. As we move to the discrete setting, a discrete Fourier transform is readily available. For reasons that will be explained, we will restrict ourselves to the simplest setting which is in finite dimensions of an odd prime number. From there, we will be able to follow a path in complete analogy with our discussion of the continuous case. The Heisenberg-Weyl group of displacements will lead us to displaced parity operators which will be used to assign values at specific phase space points to operators. In the case of finite dimensions, we will call the displaced parity operators by the name "phase point operators." A discussion of the automorphisms of the Heisenberg group enables us to determine uniqueness properties of the phase point operators. From there, analysis of the Radon transformation will bring a discussion of marginals and the meaning of lines in phase space. We will define the discrete Wigner function so
as to satisfy the correct marginal properties. Finally, we discuss how we can expand our approach to apply to prime-power dimensions via the use of finite fields.

## Section 4.2

## The Geometric Picture

As we discuss the mathematical structure below, it will be conceptually useful to start with a geometric scaffolding upon with the mathematical concepts may be layered. Although the mathematical development is somewhat lengthy, the geometrical representation of a finite-dimensional quantum object is relatively simple. Following Wootters [143], here we outline a useful geometric picture, making claims that will be put on a solid mathematical foundation below.

For the case of an odd prime number $N$, there is a beautiful duality between the objects in finite geometries and quantum mechanics. A quantum system with an $N$-dimensional Hilbert space will be represented on an $N \times N$ array of points. Due to the boundary conditions, we will see that the topology of the space is actually a toroidal lattice. We can give a more precise description and label the points $\alpha$ with coordinates $\left(a_{1}, a_{2}\right)$ by elements of the finite field $\mathbb{Z}(N)$ composed of the set of integers modulo $N$. Unless otherwise specified, throughout this chapter Greek letters denote points with numbered Latin letters their associated coordinates. An example of a discrete phase space corresponding to a five dimensional Hilbert space can be seen in Figure (4.1).

There exists some arbitrary set of $N$ orthonormal basis states for our Hilbert space which are eigenstates of some observable. Let the observable be called 'position' and we will use it to label the horizontal axis. The position eigenvalues will label the coordinates along this axis. Let us now make a correspondence between the 'lines' that appear perpendicular to this axis and the eigenstates corresponding to the axis


Figure 4.1: A discrete phase space corresponding to a five-dimensional Hilbert space is displayed. We choose to label its horizonal axis with position $q$ and the vertical axis with the Fourier conjugate momentum $p$.
coordinate. We will label the vertical axis with the Fourier conjugate observable to the 'position' which we will call 'momentum'.

Marginal properties will be of fundamental importance, as in the continuous case. We have just required that lines give geometric meaning to the label of an axis. A marginal property consists of a set of results from sums over parallel lines. In the discrete setting, a line is any set of points $\alpha=\left(a_{1}, a_{2}\right)$ such that $m a_{1}+n a_{2}=p$ with $m, n, p$ belonging to the field $\mathbb{Z}(N)$. Lines are parallel if they have the same values $m$ and $n$ with $p$ different. Each line belongs to a set of $N$ parallel lines which we call a 'foliation'. An example of a foliation for the case of $N=5$ is displayed in Fig (4.2). This is easy to count for the horizontal and vertical cases. As we will show, our correspondence between lines and eigenstates exists due to the fact that they both
appear in $(N+1)$ sets of $N$ objects.


Figure 4.2: Two foliations for a $5 \times 5$ phase space. Top row: a folation of vertical lines. Bottom row: a foliation of diagonal lines.

How many 'directions' exist in this space? Anchoring a set of lines at the origin, we can see how a set of non-parallel lines can be described by a set of elementary displacements from the origin. Using the notation for displacement as $\hat{D}_{\alpha}=\hat{D}\left(a_{1}, a_{2}\right)$ with the first entry indicating a vertical displacement and the second entry indicating a horizontal displacement, this elementary set is $\{\hat{D}(1,0), \hat{D}(0,1), \hat{D}(1,1), \hat{D}(2,1)$, $\hat{D}(3,1), \ldots \hat{D}(N, 1)\}$. Figure 4.2 shows how lines are generated by picking a single one of these displacements and iterating it. Mathematically, this is a cyclic module generated by the choice of displacement $\hat{D}\left(a_{1}, a_{2}\right)$. This brings up some questions that are useful to ponder. Can you find other choices for displacements that generate
the same set of lines? Is this the maximal set of $(N+1)$ lines through the origin? The answer to both questions is "yes" [138]. For each line in a maximal set of lines through a point, we can generate a foliation. Therefore, there exist $N(N+1)$ lines in phase space with $(N+1)$ different foliations.


Figure 4.3: Repeated action of a single displacement represented with an arrow is displayed. Note how one arrow leaves the top, only to wrap back around when we consider the points as a toroidal lattice. What other choices of displacements generate this same line?

Parenthetically, there is some disagreement in the literature about the ordering of entries in tuples. We adopt the somewhat unsatisfactory convention that the first entry, say of a point $\alpha=\left(a_{1}, a_{2}\right)$ corresponds to the vertical direction and the second to the horizontal direction.

Area is also an important concept in this space. It is defined by a set of three
ordered points $\{\alpha, \beta, \gamma\}$ that we will call an oriented triangle. The triangle $\{(0,0)$, $(0,1),(1,0)\}$ will be our standard triangle with area defined to be one (twice what we would intuitively think). A linear transformation on the space can be used to create any arbitrary triangle $\{\alpha, \beta, \gamma\}$ from the standard one. We will have a full discussions of the relevant transformations later in this chapter but for now we can provide a simple method of computing area. Calculate the area of the triangle as you normally would, multiply by 2 (convert to units of the standard area), express the result to $\bmod N$, take the additive inverse $\bmod N$ if the points are ordered clockwise. Another way to calculate this same area is to define a parallelogram by the two vectors $\beta-\alpha=\left(b_{1}-a_{1}, b_{2}-a_{2}\right)$ and $\gamma-\alpha=\left(c_{1}-a_{1}, c_{2}-a_{2}\right)$ as in Fig. (4.2). The area of such a parallelogram can be calculated as $\langle\beta-\alpha, \gamma-\alpha\rangle \bmod N$ where $\langle\cdot, \cdot\rangle$ is the symplectic form

$$
\begin{equation*}
\langle\alpha, \beta\rangle=a_{1} b_{2}-a_{2} b_{1} . \tag{4.1}
\end{equation*}
$$

In our discussion of lines, we hinted that we may be able to find different displacements to generate the same line. For the foliation displayed in the bottom row of Fig. (4.2), we see that there are four different displacements to reach each of the other four points from the origin. The displacements are $\hat{D}(2,1), \hat{D}(4,2), \hat{D}(1,3)$, and $\hat{D}(3,4)$. Can area help us understand these four different displacements that generate the same line? Notice how for any pair of displacements $\hat{D}_{\alpha}$ and $\hat{D}_{\beta}$, the area $\langle\alpha, \beta\rangle$ formed is always zero.

Returning to the task of linking physical quantities to the geometry, we can understand the labeling of axes in the continuous case as the statement that the lines perpendicular to the axis are labeled with the value at that axis. For example, the vertical lines are labelled with a position value and the horizontal lines are labelled with a momentum value. Each foliation is associated with its respective observable. There is not a natural concept of orthogonality due to the lack of a natural metric so


Figure 4.4: Two triangles $(\alpha, \beta, \gamma)$ and $\left(\alpha^{\prime}, \beta^{\prime}, \gamma^{\prime}\right)$ are shown on the $5 \times$ phase space. The first is the standard triangle with area one; the square extended from it by the dashed lines is a special case of the parallelogram constructed from vectors $\beta-\alpha$ and $\gamma-\alpha$. Triangle $\left(\alpha^{\prime}, \beta^{\prime}, \gamma^{\prime}\right)$ has area 2. Its apparent area is 3 but due to the ordering, we take the additive inverse of $3 \bmod 5$.
we do not have a physical constraint for association of a foliation to an axis. However, it is often convenient to make such a choice in both the continuous and discrete cases.

Because of the freedom we have in associating a foliation to an axis, there is a great deal of symmetry in our problem. Physical symmetries are naturally described in the language of groups. Let us think back to our example two paragraphs back. Using modular arithmetic, all the displacements can be generated from repeated action of any single one. If fact, one can check that with the addition of the trivial displacement $\hat{D}(0,0)$, these have the properties of an algebraic group. This is just an introduction to many important group theoretic properties when studying (finite dimensional)
quantum systems. In our current setting, we might expect to see the appearance of some group containing displacements and preserving areas as a reflection of the symmetry on the space upon which we will describe the physics.

The only important constraint upon the axis observables is that they must be mutually unbiased. This means that for two sets of eigenstates $\left\{\left|\psi_{m}\right\rangle\right\}$ and $\left\{\left|\phi_{n}\right\rangle\right\}$ corresponding to two different observable axes, the overlap $\left|\left\langle\psi_{m} \mid \phi_{n}\right\rangle\right|^{2}$ must be independent of $m$ and $n$. In the discrete case, this means that overlap must be $1 / N$ due to normalization. Mutual unbiasedness will be foundational to the association of a Wigner function on this space to a quantum state.

As we shall see in the mathematical development to come, certain pure states will be associated to lines in the phase space taking a value $1 / N$ at each point. The probabilistic interpretation is that for the associated observable, the probability of finding the system with an eigenvalue corresponding to the line under consideration is 1 . For any other observable corresponding to a foliation that is not parallel to the first, the probability of finding the system in any state is $1 / N$. A example for the case of the position pure state $|\hat{q} ; 3\rangle$ in the 5 -dimensional phase space is shown in Fig. (4.5).

## - Section 4.3

## Rings and Fields

Here we present the reason for restricting ourselves to only prime dimensions. It revolves around the distinction between the algebraic entities called Rings and Fields. This discussion naturally leads to an introduction to the theory of Finite Fields which can be used to extend all of our results from the regime of odd primes to odd prime powers. The mathematical material presented can be found in any textbook on abstract algebra or finite fields (e.g. [37, 86]).


Figure 4.5: A position eigenstate $|\hat{q} ; 3\rangle$ is represented as a line in the $5 \times 5$ phase space. Large circles mark positions where we can imagine a Wigner function taking the value $1 / 5$ while the small dots mark positions where the Wigner function has value 0 .

Recall that a group describes the pair $(G, \circ)$, where $G$ is some set and $\circ$ is a binary operation of the group elements $g \in G$, if and only if $G$ is closed under composition via $\circ$, the composition is associative, there exists and identity element, and there exists inverse elements. The groups we are interested in will be mostly nonabelian or noncommutative because their elements will associated with noncommuting quantum operators. Building off this definition, we can define a ring which is a set $R$ equipped with two binary operations, addition + and multiplication $\cdot$ which satisfy the following set of ring axioms:
(i) R is an abelian group under addition.
(ii) Multiplication is associative and there exists a multiplicative identity element.
(iii) Multiplication is distribution with respect to addition.

Perhaps the most familiar example of a ring is the integers $\mathbb{Z}$ with normal addition and multiplication. Note that there exist subsets of rings that are closed under the operations of multiplication and addition. The even numbers is one example found for $\mathbb{Z}$. Such subsets are called ideals and ideals that can be generated by a single element are called principle ideals.

A field is just a ring with a further constraint. We might like to introduce a set $F$ of elements so that the multiplication operation also has an inverse. Thinking to the example of the integers, we know that we will run into trouble due to the identity element for the operation addition which we call zero 0 . We therefore define a field according to these axioms:
(i) $F$ is an abelian group under addition.
(ii) The nonzero elements of $F$ form an abelian group under multiplication.
(iii) Multiplication distributes over addition.

To promote the integers $\mathbb{Z}$ from ring to field status, we must introduce the rational numbers so $\mathbb{Q}$ is a field.

The integers and the rationals are countably infinite sets. In order to tailor the ideas of rings and fields to our discussion of finite dimensional quantum systems, we must construct finite sets that obey the above axioms. Modular arithmetic is a convenient way to do this. Working with the set $\mathbb{Z}$, we define an equivalence class of integers whose difference is some integer multiple $k$ of an integer $n$ called a modulus,

$$
\begin{equation*}
a \equiv b(\bmod n) \quad \text { if and only if } \quad a-b=k n . \tag{4.2}
\end{equation*}
$$

For ease of notation, we will use an integer to denote its equivalence class. In this setting, arithmetic can be performed as one would normally using the integers with the understanding that by numbers we represent the equivalence class.

Let us look at the example of arithmetic amongst the integers $\{0,1,2,3\}$ modulo 4 which we denote $\mathbb{Z}(4)$. Is it easy to check that the group properties under addition are satisfied. Multiplication amongst our set always produces some integer that will belong to the equivalence class so we have closure. 1 is the multiplicative identity so we know we are working with a ring. In order to be a field, we must have multiplicative inverses. These are the elements that when multiplied by some other element, product the identity 1. The classes 1 and 3 can be seen to be their own inverses. However, $2 \cdot 2=0 \bmod 4$ and $2 \cdot 1=2 \cdot 3=2 \bmod 4.2$ does not have a multiplicative inverse so the integers modulo 4 does not form a field but only a ring. Note that whenever we deal with integers modulo some odd prime $N$, we will write $1 / 2$ to indicate the inverse element of $2 \bmod N$ which can be calculated as $(N+1) / 2$. One can show that only with a prime modulus will the field structure exist for the integers.

The continuous Wigner function is defined on a phase space composed of a Cartesian product of the real numbers $\mathbb{R}$ which form a field. We would like to produce a discrete Wigner function with many of the same properties so we also seek a field to describe the coordinates of our phase space. As further motivation, consider what would happen if we try to define lines on a phase space defined over $\mathbb{Z}(4)$. A line was defined as the points that provide a solution to a linear equation of the form $a q+b p=c$. Consider the equation $q+2 p=0(\bmod 4)$. A solution is $\{(0,0),(2,1),(0,2),(2,3)\}$. There also exists the equation $q=0$ which has a solution $\{(0,0),(0,1),(0,2),(0,3)\}$. These sets intersect at two points instead of one which creates a contradiction with our Euclidean notion of a line.

We have just shown one reason why we would like to restrict ourselves to only
prime dimensional systems; we want to retain certain geometrical properties that occur only in a vector space which requires being defined over a field. However, there is one way to generalize our treatment while remaining within the realm of field. This is to use finite fields which have a prime power number of elements [53, 136]. This has a natural physical interpretation as a tensor product space of some number of interacting prime dimensional subsystems or elementary particles.

We now give a brief introduction to finite fields and their salient features. This section can be ignored if the reader is happy to settle for applying our results only to the case of prime dimensions without the extension to prime powers. Let $\mathbb{F}_{N}$ denotes a finite field with $N$ elements. This is sometimes called a Galois field (and quantum systems defined over finite fields are sometimes called Galois quantum systems [137]). Then $N$ must be a prime power so that $N=p^{k}$ where $p$ is some prime and $k$ is some positive integer. The prime $p$ is called the characteristic of $\mathbb{F}_{N}$ and is the smallest integer such that

$$
\begin{equation*}
\underbrace{1+1+\cdots+1}_{p \text { times }}=0 . \tag{4.3}
\end{equation*}
$$

$\mathbb{F}_{p}$ is just $\mathbb{Z}(p)$ but $\mathbb{F}_{N}$ is clearly not $\mathbb{Z}(N)$ as we have already noticed in the case of $N=4$. To construct $\mathbb{F}_{N}$, we need a polynomial $f(x)$ of degree $k$ which is irreducible in $\mathbb{F}_{p}$. An irreducible polynomial has no divisors except for scalar multiples of itself and scalars. Defining $\alpha$ as a root of $f(x)$ (i.e. $f(\alpha)=0$ ), the field obtained by adjoining $\alpha$ to $\mathbb{F}_{p}$ is $\mathbb{F}_{N}=\mathbb{F}_{p}(\alpha) \cong \mathbb{F}_{p}[x] /(f(x))$. That is, the finite field with $N=p^{k}$ elements is congruent to the set of polynomials defined over the field $\mathbb{F}_{p}$ modulo the principle ideal denoted $(f(x))$ consisting of all multiples $g(x) f(x)$, for $g(x) \in \mathbb{F}_{p}[x]$. $\mathbb{F}_{p}[x]$ denotes the set of all polynomials in $x$ over $\mathbb{F}_{p}$. In general, different irreducible polynomials of the same degree lead to isomorphic finite fields.

Building off our example above for the ring $\mathbb{Z}(4)$, consider the finite field $\mathbb{F}_{4} \cong$
$\mathbb{F}_{2}[x] /\left(x^{2}+x+1\right)=\mathbb{F}_{2}(\alpha)=\{0,1, \alpha, \alpha+1\}$ where

$$
\begin{equation*}
\alpha^{2}+\alpha+1=0 \tag{4.4}
\end{equation*}
$$

In the field $\mathbb{F}_{2}=\{0,1\}$, there are no solutions to Eq. (4.4) and this equation is used to define $\alpha$. Irreducibility is affirmed by dividing $x^{2}+x+1$ by the lower degree polynomials $x$ and $x+1$. Every element of $\mathbb{F}_{2}(\alpha)$ has the form $a_{1} \alpha+a_{0}$, where $a_{j} \in \mathbb{F}_{2}$. The set $\{1, \alpha\}$ is therefore a vector space basis of $\mathbb{F}_{4}$ over $\mathbb{F}_{2}$.

The arithmetic is uniquely determined by Eq. (4.4). For example, we have the square of $\alpha$ as

$$
\begin{equation*}
\alpha^{2}=-\alpha-1=(-1) \alpha+(-1) 1=\alpha+1 . \tag{4.5}
\end{equation*}
$$

The square of this resulting value is

$$
\begin{equation*}
(\alpha+1)^{2}=\alpha^{2}+(1+1) \alpha+1=\alpha^{2}+1=(\alpha+1)+1=\alpha . \tag{4.6}
\end{equation*}
$$

$\alpha$ and $\alpha+1$ are each others multiplicative inverse:

$$
\begin{equation*}
\alpha(\alpha+1)=\alpha^{2}+\alpha=-1=1 . \tag{4.7}
\end{equation*}
$$

The field basis $E$ is an ordered set of $k$ field elements $\left\{e_{1}, \ldots, e_{k}\right\}$ such that every element $x$ in $\mathbb{F}_{N}$ can be expressed in the form

$$
\begin{equation*}
x=\sum_{j}^{k} x_{j} e_{j} \tag{4.8}
\end{equation*}
$$

where $x_{j}$ is always in the prime field $\mathbb{F}_{p}$. This was alluded to in the example of $\mathbb{F}_{4}$. There are in general many possible choices for a field basis.

The map $\sigma: \alpha \rightarrow \alpha^{p}$, where $\alpha \in \mathbb{F}_{N}$, is a linear automorphism of $\mathbb{F}_{N}$ called the

Frobenius automorphism leading to Galois conjugates. Elements of the prime field are invarient under action of the Frobenius automorphism. The trace operation

$$
\begin{equation*}
\operatorname{tr}(\alpha)=\alpha+\alpha^{p}+\alpha^{p^{2}}+\cdots+\alpha^{p^{k-1}}=\sum_{m=0}^{k-1} \sigma^{m}(\alpha), \tag{4.9}
\end{equation*}
$$

maps any element of the finite field in an element of the prime field: $\operatorname{tr}: \mathbb{F}_{N} \rightarrow \mathbb{F}_{p}$. Special care will be taken to distinguish the Hilbert space trace ' $\operatorname{Tr}$ ' from the field trace 'tr' with capitalization of the ' $t$ '.

For any basis $E=\left\{e_{0}, e_{1}, \ldots, e_{N-1}\right\}$, there exists a unique field basis $\tilde{E}=$ $\left\{\tilde{e}_{0}, \tilde{e}_{1}, \ldots, \tilde{e}_{N-1}\right\}$ such that $\operatorname{tr}\left(\tilde{e}_{j} e_{k}\right)=\delta_{j k} . \quad \tilde{E}$ is called the dual of $E$. The dual basis $\tilde{E}$ can be used to find the unique expansion coefficients for basis $E$ of any field element x . Taking the expansion of x , multiply both sides by $\tilde{e}_{s}$ and take the field trace:

$$
\begin{equation*}
\operatorname{tr}\left(x \tilde{e}_{s}\right)=\sum_{r}^{k} x_{r} \operatorname{tr}\left(e_{r} \tilde{e}_{s}\right)=x_{s} \tag{4.10}
\end{equation*}
$$

Additive characters of $\mathbb{F}_{N}$ are defined as

$$
\begin{align*}
& \chi(\alpha)=\omega[\operatorname{tr}(\alpha)] ; \quad \alpha \in \mathbb{F}_{N}  \tag{4.11}\\
& \omega(k)=\exp \left(\frac{2 \pi i k}{p}\right) ; \quad[\chi(\alpha)]^{p}=1 ; \quad k \in \mathbb{F}_{p} \tag{4.12}
\end{align*}
$$

This concludes a brief summary of facts we will find important in a generalization from prime dimensions to prime power dimensions for quantum systems. Most of the equations we will write down for prime dimensional systems can be easily generalized to prime powers using finite fields by replacing the root of unity $\omega$ (which is the additive character of the prime dimensional finite group) with $\chi$.

## - Section 4.4

## Fourier Transforms and Displacements

We will now lay down the structure of discrete phase space following Vourdas [138]. In the case of the continuous Wigner function, we wish to represent quantum states in the phase space of position and momentum which are quantum Fourier conjugates. When dealing with finite dimensional quantum systems, we may not have some natural position or momentum observables but we can create an analogous mathematical structure to the continuous case.

Treating our Hilbert space as an abstract quantity, we can choose an orthonormal basis that we consider to be 'position states', denoted as $|\hat{q} ; m\rangle$ with $m \in \mathbb{Z}(d)$. This notation (found in e.g. [116, 138]) indicates that this ket is the $m$-th eigenstate of the position operator which will be defined retroactively.

We can now define the discrete fourier transform:

$$
\begin{equation*}
\hat{F}=\frac{1}{\sqrt{N}} \sum_{m, n} \omega(m n)|\hat{q} ; m\rangle\langle\hat{q} ; n| ; \quad \omega(k)=\exp \left(\frac{2 \pi i k}{N}\right) . \tag{4.13}
\end{equation*}
$$

In the position basis $F$ is represented as

$$
\hat{F}=\frac{1}{\sqrt{N}}\left(\begin{array}{cccc}
1 & 1 & \cdots & 1  \tag{4.14}\\
1 & \omega(1) & \cdots & \omega(N-1) \\
\vdots & \vdots & \ddots & \vdots \\
1 & \omega(N-1) & \cdots & \omega(1)
\end{array}\right)
$$

The Fourier transform allows us to define a basis of 'momentum states' which are the Fourier duals to the position basis. Using the same notation convention as before, we
have

$$
\begin{equation*}
|\hat{p} ; m\rangle=\hat{F}|\hat{q} ; m\rangle=\frac{1}{\sqrt{N}} \sum_{n} \omega(m n)|\hat{q} ; n\rangle . \tag{4.15}
\end{equation*}
$$

Represented as vectors in the position basis, $|\hat{q} ; n\rangle$ and $|\hat{p} ; n\rangle$ are

$$
|\hat{q} ; n\rangle=\left(\begin{array}{c}
0  \tag{4.16}\\
\vdots \\
1 \\
\vdots \\
0
\end{array}\right) ; \quad|\hat{p} ; n\rangle=\left(\begin{array}{c}
1 \\
\omega(n) \\
\vdots \\
\omega[n(N-1)]
\end{array}\right) .
$$

It is straight forward to work out that repeated action of the Fourier operator on a position state gives a sequence

$$
\begin{equation*}
|\hat{q} ; m\rangle \xrightarrow{\hat{F}}|\hat{p} ; m\rangle \xrightarrow{\hat{F}}|\hat{q} ;-m\rangle \xrightarrow{\hat{F}}|\hat{p} ;-m\rangle \xrightarrow{\hat{F}}|\hat{q} ; m\rangle, \tag{4.17}
\end{equation*}
$$

which proves that $\hat{F}^{4}=\mathbb{1}$.
As promised, the operators for position $\hat{q}$ and momentum $\hat{p}$ are defined as

$$
\begin{equation*}
\hat{q}=\sum_{n=0}^{N-1} n|\hat{q} ; n\rangle\langle\hat{q} ; n| ; \quad \hat{p}=\sum_{n=0}^{N-1} n|\hat{p} ; n\rangle\langle\hat{p} ; n| . \tag{4.18}
\end{equation*}
$$

One can work out the commutator $[\hat{q}, \hat{p}]$, but due to the fact that we are interested in a discrete group, there exists no Lie algebra (no infinitesimal displacements) and the commutator is less useful for us.

However, some intuition from systems with continuous state space does carry over. By exponentiating the position and momentum operators, we can form the shift and clock operators

$$
\begin{equation*}
\hat{X}=\exp \left[-\frac{2 \pi i}{N} \hat{p}\right] ; \quad \hat{Z}=\exp \left[\frac{2 \pi i}{N} \hat{q}\right] . \tag{4.19}
\end{equation*}
$$

As an example, for a 5 -dimensional quantum system, $\hat{X}$ and $\hat{Z}$ are represented by $5 \times 5$ matrices in the position basis as

$$
\hat{X}=\left(\begin{array}{ccccc}
0 & 0 & 0 & 0 & 1  \tag{4.20}\\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{array}\right) ; \quad \hat{Z}=\left(\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & \omega(1) & 0 & 0 & 0 \\
0 & 0 & \omega(2) & 0 & 0 \\
0 & 0 & 0 & \omega(3) & 0 \\
0 & 0 & 0 & 0 & \omega(4)
\end{array}\right)
$$

The clock and shift operators will play a central role in our formalism instead of the position and momentum operators. We now flesh out some additional properties of these operators and show that they generate the discrete Heisenberg-Weyl group of displacements. Checking the action of these matrices on the position and momentum basis states shows us that they are special instances of displacement operators:

$$
\begin{align*}
& \hat{Z}^{m}|\hat{p} ; l\rangle=|\hat{p} ; l+m\rangle ; \quad \hat{Z}^{m}|\hat{q} ; l\rangle=\omega(l m)|\hat{q} ; l\rangle  \tag{4.21}\\
& \hat{X}^{n}|\hat{p} ; l\rangle=\omega(-l n)|\hat{p} ; l\rangle ; \quad \hat{X}^{n}|\hat{q} ; l\rangle=|\hat{q} ; l+n\rangle \tag{4.22}
\end{align*}
$$

where $l, m, n \in \mathbb{Z}(N)$. In the field of time-frequency analysis of signals, delays and Doppler shifts of a radar waveform may be represents by the operators $\hat{X}$ and $\hat{Z}$ [68].

Calculating the trace of these operators, we find

$$
\begin{equation*}
\operatorname{Tr}(\hat{X})=\operatorname{Tr}(\hat{Z})=0 \tag{4.23}
\end{equation*}
$$

Upon checking the commutation rules, we arrive at the discrete analogue of the continuous commutation relations:

$$
\begin{equation*}
\hat{X}^{n} \hat{Z}^{m}=\hat{Z}^{m} \hat{X}^{n} \omega(-m n) . \tag{4.24}
\end{equation*}
$$

The symmetry between this equation and the continuous case was what led Weyl to investigate this formulation as a toy model for the continuous case [141]. Schwinger noted the symmetry between the $\hat{X}$ and $\hat{Z}$ operators in that the commutation relation wouldn't change with the substitution $\hat{X} \rightarrow \hat{Z}$ and $\hat{Z} \rightarrow \hat{X}^{-1}$. This symmetry is due to the rooting of our theory in the Fourier transform:

$$
\begin{equation*}
\hat{F} \hat{X} \hat{F}^{\dagger}=\hat{Z} \quad \text { and } \quad \hat{F} \hat{Z} \hat{F}^{\dagger}=\hat{X}^{\dagger} . \tag{4.25}
\end{equation*}
$$

Additionally, we can show that

$$
\begin{equation*}
\hat{X}^{N}=\hat{Z}^{N}=\mathbb{1} . \tag{4.26}
\end{equation*}
$$

Thus far, we have constructed discrete analogies to all the important quantities comprising the continuous Weyl algebra and Canonical Commutation Relations. However, one distinct difference has just reared its head in Eq. (4.26), the cyclic property of the discrete setting. This will play out in many interesting ways as we go on so stay alert for situations that will probe the cyclicity of the system. For now, we can take this relation to define the topology of our phase space. $N$ steps in either the horizontal direction of position or the vertical direction of momentum lead us back to the same point in the phase space. It therefore has the structure of a $\mathbb{Z}(N) \times \mathbb{Z}(N)$ toroidal lattice.

In analogy with the geometrical discussion above, general displacement operators can now be constructed for the quantum system as the unitary operators

$$
\begin{equation*}
\hat{D}_{\alpha} \equiv \hat{D}\left(a_{1}, a_{2}\right)=\omega\left(-2^{-1} a_{1} a_{2}\right) \hat{Z}^{a_{1}} \hat{X}^{a_{2}} . \tag{4.27}
\end{equation*}
$$

We have introduced the two equivalent notations to emphasize that $\alpha$ can be un-
derstood as a 'vector' with components $\left(a_{1}, a_{2}\right)$. We must be careful with the word vector because we will understand later that the dimension $N$ will determine whether these can be thought of as living on a veritable vector space or a module.

The displacement operators satisfy the composition rules

$$
\begin{align*}
\hat{D}\left(a_{1}, a_{2}\right) \hat{D}\left(b_{1}, b_{2}\right) & =\omega\left(a_{2} b_{1}-a_{1} b_{2}\right) \hat{D}\left(b_{1}, b_{2}\right) \hat{D}\left(a_{1}, a_{2}\right)  \tag{4.28}\\
& =\omega\left[\frac{1}{2}\left(a_{2} b_{1}-a_{1} b_{2}\right)\right] \hat{D}\left(a_{1}+b_{1}, a_{2}+b_{2}\right) \tag{4.29}
\end{align*}
$$

In the vector notation, we write

$$
\begin{equation*}
\hat{D}_{\alpha} \hat{D}_{\beta}=\omega(\langle\alpha, \beta\rangle) \hat{D}_{\beta} \hat{D}_{\alpha}=\omega\left(\frac{1}{2}\langle\alpha, \beta\rangle\right) \hat{D}_{\alpha+\beta} \tag{4.30}
\end{equation*}
$$

where the object $\langle\cdot, \cdot\rangle$ in the exponent is the symplectic form.
Action on the position and momentum eigenstates are given as

$$
\begin{align*}
& \hat{D}\left(a_{1}, a_{2}\right)|\hat{x} ; j\rangle=\omega\left(2^{-1} a_{1} a_{2}+a_{1} j\right)\left|\hat{x} ; j+a_{2}\right\rangle  \tag{4.31}\\
& \hat{D}\left(a_{1}, a_{2}\right)|\hat{p} ; j\rangle=\omega\left(-2^{-1} a_{1} a_{2}+a_{2} j\right)\left|\hat{p} ; j+a_{1}\right\rangle \tag{4.32}
\end{align*}
$$

The displacement operators have the pleasing property

$$
\begin{equation*}
\hat{D}\left(a_{1}, a_{2}\right)^{-1}=\hat{D}\left(a_{1}, a_{2}\right)^{\dagger}=\hat{D}\left(-a_{1},-a_{2}\right) \tag{4.33}
\end{equation*}
$$

They are also all represented by traceless matrices except for the identity:

$$
\begin{equation*}
\operatorname{Tr} \hat{D}_{a_{1}, a_{2}}=\omega^{\frac{1}{2} a_{1} a_{2}} \sum_{j=0}^{N-1} \omega^{j a_{2}}\left\langle j \mid j+a_{1}\right\rangle=N \delta_{a_{1}, 0} \delta_{a_{2}, 0} \tag{4.34}
\end{equation*}
$$

Schwinger first identified these operators as forming a complete orthonormal basis
with $N^{2}$ elements via the Hilbert-Schmidt (trace) product. We can check

$$
\begin{equation*}
\operatorname{Tr}\left[\hat{D}_{\alpha}^{\dagger} \hat{D}_{\beta}\right]=N \delta_{\alpha, \beta} \tag{4.35}
\end{equation*}
$$

We can therefore decompose any complex operator $\hat{O}$ on $\mathcal{H}_{N}$ uniquely as

$$
\begin{equation*}
\hat{O}=\sum_{a_{1}=0}^{N-1} \sum_{a_{2}=0}^{N-1} O_{a_{1}, a_{2}} \hat{D}_{a_{1}, a_{2}}, \quad \text { or } \quad \hat{O}=\sum_{\alpha} O_{\alpha} \hat{D}_{\alpha} \tag{4.36}
\end{equation*}
$$

where summations with unlabelled range are taken over the complete space. The expansion coefficients are the complex numbers given by

$$
\begin{equation*}
O_{\alpha}=O_{a_{1}, a_{2}}=\frac{1}{N} \operatorname{Tr}\left[\hat{D}_{\alpha}^{\dagger} \hat{O}\right]=\frac{1}{N} \operatorname{Tr}\left[\hat{D}_{-\alpha} \hat{O}\right] . \tag{4.37}
\end{equation*}
$$

An expansion of an arbitrary Hilbert space operator over a basis of $N^{2}$ elements is indeed what we are searching for in order to define a discrete Wigner function. However, in order to guarantee marginal properties akin the the continuous Wigner function, we must work a bit harder. However, there is an obvious path forward via the definition of the parity and displaced parity operator. Before we continue on, it is worthwhile to step back to gather some group theoretic facts that we have been weaving between into a cohesive picture.

## Section 4.5

## The Heisenberg-Weyl Group

Group theory plays a central role in the study of symmetries of a quantum system [63]. We have already hinted at this fact with the example of displacements along lines in phase space. For such a symmetry, there exists a unitary, irreducible, (possibly) projective representation of the symmetry group. The symmetry group of translational
and boost invariance of a free quantized mass point at motion in $\mathbb{R}^{n}$ is the wellknown Galilei symmetry $[67] . \mathbb{R}^{2 n}$ is the associated classical symmetry group whose unitary irreducible projective representation is given by the Weyl representation and the canonical commutation relations. The Weyl representation can be understood abstractly as a group on its own called the Heisenberg group.

For discrete systems, shift and clock operators play the role of continuous translations and boosts, respectively. In this section we will discuss how the displacement operators $\hat{D}_{\alpha}$ form a unitary operator basis with the particularly nice structure of being an algebraic group. In particular, they form a representation of the HeisenbergWeyl group $H W[\mathbb{Z}(N)]$ modulo its center (the collineation group). We now illustrate this connection starting from the abstract construction of the group at hand.

The Heisenberg-Weyl group has elements $(p, q, t)$ where $p, q$, and $t$ are elements of some field not of characteristic two. It is defined abstractly by its composition law

$$
\begin{equation*}
\left(p_{1}, q_{1}, t_{1}\right)\left(p_{2}, q_{2}, t_{2}\right)=\left(p_{1}+p_{2}, q_{1}+q_{2}, t_{1}+t_{2}+2^{-1}\left\langle\left(p_{1}, q_{1}\right),\left(p_{2}, q_{2}\right)\right\rangle\right) \tag{4.38}
\end{equation*}
$$

with the symplectic form playing a prominent role. The important commutation relation corresponding to this product is given by

$$
\begin{equation*}
\left(p_{1}, q_{1}, t_{1}\right)\left(p_{2}, q_{2}, t_{2}\right)=\left(p_{2}, q_{2}, t_{2}\right)\left(p_{1}, q_{1}, t_{1}\right)\left(0,0,\left\langle\left(p_{1}, q_{1}\right),\left(p_{2}, q_{2}\right)\right\rangle\right) \tag{4.39}
\end{equation*}
$$

For an odd prime $N$, we have already naturally defined the generators necessary to construct a representation of $H W[\mathbb{Z}(N)]$ using the clock operator $\hat{Z}$, the shift operator $\hat{X}$, and the $N$-th root of unity $\omega$. A faithful representation of the $N^{3}$ elements is provided by an combination $\omega^{t} \hat{X}^{q} \hat{Z}^{p}$.

What does this all have to do with a quantum system described in a Hilbert space $\mathcal{H}$ with dimension $N$ ? All the complex operators that act on $\mathcal{H}$ live in the

Hilbert-Schmidt space of dimension $N^{2}$ canonically isomorphic to the Hilbert space $\mathcal{H} \otimes \mathcal{H}^{*}$. We have seen that the $N^{2}$ unitary displacement operators $\hat{D}_{\alpha}$ form a complete orthonormal basis of this space. We now can show that they have an additional group property.

In fact, the operators $\hat{D}(q, p) \omega(t)$ form a faithful representation of the Heisenberg Weyl group which we will now prove. We have already looked at the composition rules for the displacement operators in Eq. (4.29). Closure of the operators $\hat{D}(q, p) \omega(t)$ is based on the commutation rule in Eq. (4.24). The identity operator exists as $\hat{D}(0,0) \omega(0)$, in inverse of $\hat{D}(q, p) \omega(t)$ exists as $\hat{D}(-q,-p) \omega(-t)$, and associativity holds. Therefore the elements $\hat{D}(q, p) \omega(t)$ form a group with composition law given in Eq. (4.38) which defines the Heisenberg-Weyl group.

The phase factors commute with all other elements of the group so they belong to the center of the group. If we ignore all the phase factors, we then have a group of $N^{2}$ elements called the collineation group of $H W \mathbb{Z}(N)$. Unitary operator bases with a group structure are called unitary operator bases of group type or nice error bases. The latter of the two names has its origins in the theory of quantum computation where they can be used to discretize errors, rendering the errors correctable [55, 21].

The $N=2$ Case
We now discuss the familiar example of qubits which also brings up a serious complication. In two dimensions, $\hat{X}$ and $\hat{Z}$ are the Pauli matrices $\sigma_{x}$ and $\sigma_{z}$, respectively. Note that $\sigma_{x}{ }^{2}=\sigma_{z}^{2}=\mathbb{1}$ while $\left(\sigma_{x} \sigma_{z}\right)^{2}=\left(i \sigma_{y}\right)^{2}=-\mathbb{1}$. We have an inconsistency in the order of the group elements. Stepping back, we can see that $\operatorname{det} \hat{Z}=\operatorname{det} \hat{X}=(-1)^{N+1}$. If $N$ is odd, the Heisenberg-Weyl group will be a subgroup of $S U(N)$. However, when $N$ is even, it is only a subgroup of $U(N)$. This means the for $N$ odd, the $N$-th power of each group element yields the identity, while for $N$ even this is only true for the $2 N$-th power of each group element. Although there are ways to handle the even case
[3, 4], it adds complications that can be distracting for the time being. In order to illustrate the main results in this thesis, we will restrict ourselves to odd prime or prime power dimensions.
$\ulcorner$ Section 4.6

## Symplectic Transformations

The symplectic form $\langle\cdot, \cdot\rangle$ is readily apparent in the structure of the HeisenbergWeyl group as seen in Eq. (4.38). This object is also central to the formulation of Hamiltonian mechanics and continuous quantum mechanics in phase space. In order to construct a satisfactory discrete phase space equipped with a symplectic structure, we require that our $N^{2}$ phase point operator basis must be transformed into itself upon action by the automorphism group of the displacement operators from operators that preserve the symplectic form.

The Clifford group $C(N)$ consists of the unitary operators $U$ that map displacement operators $\hat{D}_{\alpha}$ to other displacement operators under conjugation

$$
\begin{equation*}
\hat{U} \hat{D}_{\alpha} \hat{U}^{-1}=\omega(\phi) \hat{D}_{f(\alpha)} \tag{4.40}
\end{equation*}
$$

where $\phi$ is some irrelevant phase factor and $f(\cdot)$ is just some function. A simple example of such a unitary is just any displacement operator. Operators $\hat{U}$ permute the members of the Heisenberg-Weyl group so the Clifford group is called the normalizer of the Heisenberg-Weyl group.

With some effort [3], one can show that the function $f(\cdot)$ is linear so that the transformation can be written

$$
\begin{equation*}
\hat{S}_{G} \hat{D}_{\alpha} \hat{S}_{G}^{-1}=\hat{D}_{G \alpha}, \tag{4.41}
\end{equation*}
$$

where $G$ is a two-by-two matrix defined over the field $\mathbb{Z}(N)$ and $\hat{S}_{G}$ is the $N$ unitary representation of $G$. The symplectic form in the Heisenberg-Weyl group law (4.30) now restricts the form of matrices $G$. We know that

$$
\begin{equation*}
\hat{S}_{G} \hat{D}_{\alpha} \hat{D}_{\beta} \hat{S}_{G}^{-1}=\omega\left(\frac{1}{2}\langle\alpha, \beta\rangle\right) \hat{S}_{G} \hat{D}_{\alpha+\beta} \hat{S}_{G}^{-1} \tag{4.42}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\hat{D}_{G \alpha} \hat{D}_{G \beta}=\omega\left(\frac{1}{2}\langle\alpha, \beta\rangle\right) \hat{D}_{G(\alpha+\beta)} . \tag{4.43}
\end{equation*}
$$

However, straightforward calculation of the left hand side of the equation above shows

$$
\begin{equation*}
\hat{D}_{G \alpha} \hat{D}_{G \beta}=\omega\left(\frac{1}{2}\langle G \alpha, G \beta\rangle\right) \hat{D}_{G(\alpha+\beta)} \tag{4.44}
\end{equation*}
$$

For the last two equations to be consistent, $G$ must preserve the symplectic form

$$
\begin{equation*}
\langle\alpha, \beta\rangle=\langle G \alpha, G \beta\rangle \bmod N \tag{4.45}
\end{equation*}
$$

$\langle\alpha, \beta\rangle$ can be written explicitly as a matrix equation

$$
\langle\alpha, \beta\rangle=\left(\begin{array}{ll}
a_{1} & a_{2}
\end{array}\right)\left(\begin{array}{cc}
0 & -1  \tag{4.46}\\
1 & 0
\end{array}\right)\binom{b_{1}}{b_{2}} .
$$

For Eq. (4.45) to hold, with

$$
G=\left(\begin{array}{cc}
k & l  \tag{4.47}\\
m & n
\end{array}\right) \equiv G(k, l \mid m, n)
$$

it must be that

$$
\left(\begin{array}{cc}
0 & -1  \tag{4.48}\\
1 & 0
\end{array}\right)=\left(\begin{array}{cc}
0 & -k n+l m \\
k n-l m & 0
\end{array}\right) .
$$

$G$ must have a unit determinant (modulo $N$ ).
Such matrices form the symplectic group $S p[2, \mathbb{Z}(N)]$ and leave the symplectic form invariant. The group $S p[2, \mathbb{Z}(N)]$ is only a subgroup of the Clifford group. However, it can be shown that the Clifford group modulo its center, $C(N) / I(N)$, is equal to the semi-direct product of the symplectic rotations $S p[2, \mathbb{Z}(N)]$ and the translations $H W[\mathbb{Z}(N)]$ modulo its center [45,59]. This is a reflection of the symmetry we anticipated in the discussion of the geometry of phase space in section (4.2). The centers of these two groups, those group elements that commute with all others, are just complex phases. We are considering quantum states where global phases do not effect the physics and the groups under discussion will play a role in constructing orthonormal bases for the Hilbert-Schmidt space so it is natural to consider an equivalence up to a phase.

The symplectic group $S p[2, \mathbb{Z}(N)]$ has inverse elements [136]

$$
\left(\begin{array}{ll}
k & l  \tag{4.49}\\
m & n
\end{array}\right)^{-1}=\left(\begin{array}{cc}
n & -l \\
-m & k
\end{array}\right)
$$

The group multiplication can be written as

$$
\left(\begin{array}{ll}
k_{2} & l_{2}  \tag{4.50}\\
m_{2} & n_{2}
\end{array}\right)\left(\begin{array}{ll}
k_{1} & l_{1} \\
m_{1} & n_{1}
\end{array}\right)=\left(\begin{array}{ll}
k & l \\
m & n
\end{array}\right)
$$

where

$$
\begin{align*}
& k=k_{2} k_{1}+l_{2} m_{1}, \quad l=k_{2} l_{1}+l_{2} n_{1},  \tag{4.51}\\
& m=m_{2} k_{1}+n_{2} m_{1}, \quad n=m_{2} l_{1}+n_{2} n_{1}
\end{align*}
$$

What is left is to construct the unitary $N \times N$ representations $\hat{S}_{G}$ of the matrices $G$. Let $\hat{S}_{G}=\hat{S}(k, l \mid m, n)$ be the matrices with the following properties:

$$
\begin{align*}
& \hat{X}(k, l)=\hat{S}(k, l \mid m, n) \hat{X}[\hat{S}(k, l \mid m, n)]^{\dagger}=\omega\left(2^{-1} k l\right) \hat{X}^{k} \hat{Z}^{l}=\hat{D}(k, l),  \tag{4.52}\\
& \hat{Z}(m, n)=\hat{S}(k, l \mid m, n) \hat{Z} \hat{S}(k, l \mid m, n)^{\dagger}=\omega\left(2^{-1} m n\right) \hat{X}^{m} \hat{Z}^{n}=\hat{D}(m, n), \tag{4.53}
\end{align*}
$$

where $k n-l m=1(\bmod (N))$ and $k, l, m, n$ are integers in $\mathbb{Z}(N)$. We will explicitly construct operators $\hat{S}_{G}$ below, thus proving their existence. Due to the constraint, there are only three free parameters of these operators. The simplest examples of these matrices are

$$
\begin{equation*}
\hat{S}(1,0 \mid 0,1)=\mathbb{1}=\hat{F}^{4} ; \quad \hat{S}(0,1 \mid-1,0)=\hat{F} \tag{4.54}
\end{equation*}
$$

The transformed $\hat{X}(k, l)$ and $\hat{Z}(m, n)$ operators each depend on two of the four parameters that determines $\hat{S}(k, l \mid m, n)$ as is clear from the notation. This carries over to their eigenstates as well,

$$
\begin{align*}
& |\hat{q}(m, n) ; j\rangle=\hat{S}(k, l \mid m, n)|\hat{q} ; j\rangle  \tag{4.55}\\
& |\hat{p}(k, l) ; j\rangle=\hat{S}(k, l \mid m, n)|\hat{p} ; j\rangle \tag{4.56}
\end{align*}
$$

The cases that we are already familiar with include

$$
\begin{align*}
& |\hat{q}(0,1) ; j\rangle=|\hat{q} ; j\rangle \\
& |\hat{q}(-1,0) ; j\rangle=|\hat{p} ; j\rangle  \tag{4.57}\\
& |\hat{p}(1,0) ; j\rangle=|\hat{p} ; j\rangle, \\
& |\hat{p}(0,1) ; j\rangle=|\hat{q} ;-j\rangle .
\end{align*}
$$

We now sketch the proof the matrices $\hat{S}_{G}=\hat{S}(k, l \mid m, n)$ form a representation of $S p[2, \mathbb{Z}(N)]$. Using Eq. (4.52) with both $\hat{S}\left(k_{1}, l_{1} \mid m_{1}, n_{1}\right)$ and $\hat{S}\left(k_{2}, l_{2} \mid m_{2}, n_{2}\right)$, we can prove closure of their product with the analogous result to Eq. (4.50) with $\hat{S}_{G}$ corresponding to $G$. Therefore, the $\hat{S}(k, l \mid m, n)$ matrices form a $N \times N$ representation we sought with group multiplications rule

$$
\begin{equation*}
\hat{S}\left(k_{1}, l_{1} \mid m_{1}, n_{1}\right) \hat{S}\left(k_{2}, l_{2} \mid m_{2}, n_{2}\right)=\hat{S}(k, l \mid m, n) \tag{4.58}
\end{equation*}
$$

where $k, l, m, n$ are just as in Eq. (4.51).

## Section 4.7

## Phase Point Operators

Having studied the Heisenberg-Weyl group and its symplectic transformation, we return to our development of the discrete Wigner function. Now we can formulate the discrete version of the displaced parity operators which we call phase point operators. As it has been, our discussion is in complete analogy with the continuous case covering parity operators, displaced parity operators, and marginal properties.

The parity operator (at the origin) is defined as the square of the discrete Fourier transform

$$
\begin{equation*}
\hat{P} \equiv \hat{P}(0,0)=\hat{F}^{2} \tag{4.59}
\end{equation*}
$$

About a point $\alpha$, the phase point operator is

$$
\begin{equation*}
\hat{A}_{\alpha}=\hat{D}_{\alpha} \hat{P} \hat{D}_{\alpha}^{-1} . \tag{4.60}
\end{equation*}
$$

We denote the phase point operators at point $\alpha$ as $\hat{A}_{\alpha}$ to conform with Wootters' notation $[143,53]$. Note that

$$
\begin{equation*}
\hat{A}_{\alpha}^{2}=\mathbb{1}=\hat{P}^{2} \tag{4.61}
\end{equation*}
$$

and both $\hat{A}_{\alpha}$ and $\hat{P}$ are unitary. The eigenvalues of the phase point operators are $\pm 1$, so the operators are both Hermitian and unitary.

We now list some properties of the parity operator that can are all straightforward to prove from the action of the discrete Fourier transform in Eq. (4.17) and the displacement operators in Eq. (4.31) on the basis operators:

$$
\begin{align*}
& \hat{A}\left(a_{1}, a_{2}\right)|\hat{q} ; j\rangle=\omega\left(2 a_{1} a_{2}-2 a_{1} j\right)\left|\hat{q} ; 2 a_{2}-j\right\rangle  \tag{4.62}\\
& \hat{A}\left(a_{1}, a_{2}\right)|\hat{p} ; j\rangle=\omega\left(-2 a_{1} a_{2}-2 a_{2} j\right)\left|\hat{q} ; 2 a_{2}-j\right\rangle \\
& \hat{P} \hat{D}\left(a_{1}, a_{2}\right) \hat{P}=\hat{D}\left(-a_{1},-a_{2}\right)  \tag{4.63}\\
& \hat{A}\left(a_{1}, a_{2}\right) \hat{A}\left(b_{1}, b_{2}\right)=\hat{D}\left(2 a_{1}-2 b_{1}, 2 a_{2}-2 b_{2}\right) \omega\left(2 a_{2} b_{1}-2 a_{1} b_{2}\right),  \tag{4.64}\\
& \hat{S}(k, l \mid m, n) \hat{A}\left(a_{1}, a_{2}\right) \hat{S}(k, l \mid m, n)^{\dagger}=\hat{A}\left(a_{1} n+a_{2} l, a_{1} m+a_{2} k\right) \tag{4.65}
\end{align*}
$$

We are familiar with the important set of marginal properties of continuous distributions. There are similar marginal properties resulting from sums over lines in discrete phase space. We can find the simplest marginals in the 'horizontal' or 'position' and 'vertical' or 'momentum' direction for both the displacement and parity operators. Let $\hat{\Pi}_{q}(j)$ and $\hat{\Pi}_{p}(j)$ be projectors onto position and momentum states so
that

$$
\begin{equation*}
\hat{\Pi}_{q}(j)=|\hat{q} ; j\rangle\langle\hat{q} ; j| ; \quad \hat{\Pi}_{p}(j)=|\hat{p} ; j\rangle\langle\hat{p} ; j| . \tag{4.66}
\end{equation*}
$$

The marginal properties for the displacement operators are

$$
\begin{align*}
& \frac{1}{N} \sum_{a_{2}=0}^{N-1} \hat{D}\left(a_{1}, a_{2}\right)=\hat{\Pi}_{p}\left(2^{-1} a_{1}\right) \hat{A}(0,0) \\
& \frac{1}{N} \sum_{a_{1}=0}^{N-1} \hat{D}\left(a_{1}, a_{2}\right)=\hat{\Pi}_{q}\left(a_{2}\right) \hat{A}(0,0)  \tag{4.67}\\
& \frac{1}{N} \sum_{a_{1}, a_{2}} \hat{D}\left(a_{1}, a_{2}\right)=\hat{A}(0,0)
\end{align*}
$$

The first two equations can be proven by taking $\langle\hat{q} ; m|$ one the left side and $|\hat{q} ; n\rangle$ on the right. The third falls into place with a proper summation in either the first or second equation. For the phase point operators, we have

$$
\begin{align*}
& \frac{1}{N} \sum_{a_{2}=0}^{N-1} \hat{A}\left(a_{1}, a_{2}\right)=\hat{\Pi}_{p}\left(a_{1}\right) \\
& \frac{1}{N} \sum_{a_{1}=0}^{N-1} \hat{A}\left(a_{1}, a_{2}\right)=\hat{\Pi}_{q}\left(a_{2}\right)  \tag{4.68}\\
& \frac{1}{N} \sum_{a_{1}, a_{2}} \hat{A}\left(a_{1}, a_{2}\right)=\mathbb{1}
\end{align*}
$$

Multiplication of Eqs. (4.67) by $\hat{A}(0,0)$ yields these three relations.
We can now exhibit a very nice relationship between the displacement operators and the phase point operators; they are connected via Fourier transforms

$$
\begin{align*}
\frac{1}{N} \sum_{a_{1}, a_{2}} \hat{D}\left(a_{1}, a_{2}\right) \omega\left(a_{2} b_{1}-a_{1} b_{2}\right) & =\hat{A}\left(b_{1}, b_{2}\right) \\
& \Uparrow  \tag{4.69}\\
\frac{1}{N} \sum_{\alpha} \hat{D}_{\alpha} \omega^{\langle\beta, \alpha\rangle} & =\hat{A}_{\beta}
\end{align*}
$$

The inverse Fourier transform gives

$$
\begin{equation*}
\frac{1}{N} \sum_{\beta} \hat{A}_{\beta} \omega^{\langle\alpha, \beta\rangle}=\hat{D}_{\alpha} \tag{4.70}
\end{equation*}
$$

Eq. (4.69) follows from applying the displacement operator multiplication rule Eq. (4.30) after multiplication of Eq. (4.67) by $\hat{D}\left(b_{1}, b_{2}\right)$ on the left and $\hat{D}\left(b_{1}, b_{2}\right)^{\dagger}$ on the right. In practice, it can be useful to express the phase point operators in the form of the clock and shift operators by representing the displacement operator in their form:

$$
\begin{equation*}
\hat{A}_{\alpha}=\hat{A}\left(a_{1}, a_{2}\right)=\frac{1}{N} \sum_{b_{1}, b_{2}} \omega^{a_{2} b_{1}-a_{1} b_{2}} \omega^{b_{1} b_{2} / 2} \hat{X}^{b_{1}} \hat{Z}^{b_{2}} \tag{4.71}
\end{equation*}
$$

We have only displayed marginal properties in horizontal and vertical directions. A complete set of marginals along all other axes can be found by applying symplectic transformations to the starting two. This is the discrete Radon transform which we will write down along with its inverse. As in the continuous case, these provide the crucial connection between the measurements of observables we can take and reconstruction of a distribution over the space. This process is called quantum tomography.

Symplectic transformations of the projectors $\hat{\Pi}_{q}(j)$ onto position and $\hat{\Pi}_{p}(j)$ onto momentum yield

$$
\begin{align*}
\hat{\Pi}_{q}^{(m, n)}(j) & =\hat{S}(k, l \mid m, n) \hat{\Pi}_{q}(j) \hat{S}(k, l \mid m, n)^{\dagger}  \tag{4.72}\\
& =|\hat{q}(m, n) ; j\rangle\langle\hat{q}(m, n) ; j|
\end{align*}
$$

and

$$
\begin{align*}
\hat{\Pi}_{p}^{(k, l)}(j) & =\hat{S}(k, l \mid m, n) \hat{\Pi}_{p}(j) \hat{S}(k, l \mid m, n)^{\dagger}  \tag{4.73}\\
& =|\hat{p}(k, l) ; j\rangle\langle\hat{p}(k, l) ; j|
\end{align*}
$$

Performing symplectic transformations on each side of Eq. (4.68) and then applying Eq. (4.65), we arrive at the Radon transform:

$$
\begin{align*}
& \frac{1}{N} \sum_{a_{2}} \hat{A}\left(a_{1} n+a_{2} l, a_{1} m+a_{2} k\right)=\hat{\Pi}_{p}^{(k, l)}\left(a_{1}\right) \\
& \frac{1}{N} \sum_{a_{1}} \hat{A}\left(a_{1} n+a_{2} l, a_{1} m+a_{2} k\right)=\hat{\Pi}_{q}^{(m, n)}\left(a_{2}\right) \tag{4.74}
\end{align*}
$$

Its inverse transformation is

$$
\begin{align*}
\hat{D}\left(l a_{2}, k a_{2}\right) & =\sum_{a_{1}} \hat{\Pi}_{p}^{(k, l)}\left(a_{1}\right) \omega\left(-a_{1} a_{2}\right),  \tag{4.75}\\
\hat{D}\left(n a_{2}, m a_{2}\right) & =\sum_{a_{2}} \hat{\Pi}_{q}^{(m, n)}\left(a_{2}\right) \omega\left(a_{1} a_{2}\right)
\end{align*}
$$

Notice how the summations in Eqs. (4.74) lead to a stepping along a line in phase space with slope determined by either the pair $(k, l)$ or $(m, n)$.

## - Section 4.8 <br> Discrete Wigner and Weyl Functions

As we have seen in the first half of the thesis, the Wigner function plays an essential role as the representation of quantum mechanical states in phase space. It is a natural subject to study both for the geometric information it conveys and also as it is directly connected to measurable data.

For an arbitrary operator $\hat{O}$, the Wigner function and its cousin the Weyl function are respectively defined as

$$
\begin{equation*}
O_{\alpha} \equiv \operatorname{Tr}\left[\hat{O} \hat{A}_{\alpha}\right] ; \quad \tilde{O}_{\alpha} \equiv \operatorname{Tr}\left[\hat{O} \hat{D}_{\alpha}\right] \tag{4.76}
\end{equation*}
$$

This completes the analogy between the continuous and discrete Wigner functions.

They can both be interpreted as quasiprobability distribution that may take negative values. Again, because of the role of the displaced parity operator as the phase point operator, an interpretation of what the Wigner function actually plots is a quantification of the reflection symmetry are a point; it is the overlap between a state and its reflection around a point. In the continuous case, we arrived at that understanding from the vantage of the marginal probability distributions which we will soon study.

The Weyl function is a generalized correlation function. For $\hat{O}=|\psi\rangle\langle\psi|$, the Weyl function is the overlap of $|\psi\rangle$ and $\hat{D}_{\alpha}|\psi\rangle$. Instead of capturing the reflection symmetry of a representation of an operator on phase space as the Wigner function does, the Weyl function captures the translational symmetry under a displacement $\hat{D}_{\alpha}$.

We now show how an expansion of an arbitrary operator $\hat{O}$ can be accomplished over the displacement operators with Weyl functions as coefficients or over the phase point operators with Wigner functions as coefficients. Start by defining the matrix elements in the position or momentum bases of $\hat{O}$ as

$$
\begin{equation*}
O_{q}(r, s) \equiv\langle\hat{q} ; r| \hat{O}|\hat{q} ; s\rangle ; \quad O_{p}(r, s) \equiv\langle\hat{p} ; r| \hat{O}|\hat{p} ; s\rangle \tag{4.77}
\end{equation*}
$$

Here are two useful Fourier transform relationships for the Wigner and Weyl functions:

$$
\begin{align*}
O\left(a_{1}, a_{2}\right) & =\sum_{j} \omega\left(2 a_{1} j\right) O_{q}\left(a_{2}-j, a_{2}+j\right)  \tag{4.78}\\
& =\sum_{j} \omega\left(2 a_{2} j\right) O_{p}\left(a_{1}-j, a_{1}+j\right)
\end{align*}
$$

$$
\begin{align*}
\tilde{O}\left(a_{1}, a_{2}\right) & =\sum_{j} \omega\left(a_{1} j\right) O_{q}\left(j-2^{-1} a_{2}, j+2^{-1} a_{2}\right)  \tag{4.79}\\
& =\sum_{j} \omega\left(-a_{2} j\right) O_{p}\left(j-2^{-1} a_{1}, j+2^{-1} a_{1}\right) .
\end{align*}
$$

These can be proved by writing out the trace in Eq. (4.76) in position or momentum bases and then accounting for the action of the phase point operator and displacement operator on the resplective bases as in Eqs. (4.62) and (4.31). There is also the symplectic Fourier transform relationship between the Weyl and Wigner functions

$$
\begin{equation*}
\tilde{O}_{\alpha}=\frac{1}{N} \sum_{\beta} O_{\beta} \omega(\langle\alpha, \beta\rangle) \tag{4.80}
\end{equation*}
$$

obtained by multiplying both sides of Eq. (4.69) by $\hat{O}$ and then taking the trace. Our expansion of operator $\hat{O}$ can now be written as

$$
\begin{equation*}
\hat{O}=\frac{1}{N}=\sum_{\alpha} \tilde{O}_{-\alpha} \hat{D}_{\alpha}=\frac{1}{N} \sum_{\alpha} O_{\alpha} \hat{A}_{\alpha} \tag{4.81}
\end{equation*}
$$

Take matrix elements in position and momentum bases on both sides and in Eq. (4.77) and then use the Fourier transform relationships that follow that equation.

Finally, let us connect the mathematics to the physical quantities which we might measure. The Radon transform is used in quantum tomography and provides the bridge between the Wigner and Weyl functions and probabilities which can be measured experimentally. For a density matrix $\hat{\rho}$, the $\operatorname{Tr}\left[\hat{\rho} \hat{\Pi}_{p}^{(k, l)}(j)\right]$ are probabilities of being found in some general basis state $|\hat{p}(k, l) ; j\rangle$. By altering $(k, l)$, we can build up the full set of statistics necessary to determine the quantum state of the system (i.e. either the Wigner function or the density operator). Although we keep the development general, hold the example of $\hat{O}=\hat{\rho}$ in mind for what follows.

The simple 'horizontal' and 'vertical' marginal properties of the Wigner and Weyl
functions follow directly from Eqs. (4.68) and (4.67) by multiplying each by $\hat{O}$ and taking the trace. We have

$$
\begin{align*}
& \frac{1}{N} \sum_{a_{2}=0}^{N-1} \tilde{O}\left(a_{1}, a_{2}\right)=O_{p}\left(-2^{-1} a_{1}, 2^{-1} a_{1}\right) \\
& \frac{1}{N} \sum_{a_{1}=0}^{N-1} \tilde{O}\left(a_{1}, a_{2}\right)=O_{q}\left(-2^{-1} a_{2}, 2^{-1} a_{2}\right)  \tag{4.82}\\
& \frac{1}{N} \sum_{a_{1}, a_{2}=0} \tilde{O}\left(a_{1}, a_{2}\right)=O(0,0)
\end{align*}
$$

and

$$
\begin{align*}
& \frac{1}{N} \sum_{a_{2}=0}^{N-1} \tilde{O}\left(a_{1}, a_{2}\right)=O_{p}\left(-2^{-1} a_{1}, 2^{-1} a_{1}\right) \\
& \frac{1}{N} \sum_{a_{1}=0}^{N-1} \tilde{O}\left(a_{1}, a_{2}\right)=O_{q}\left(-2^{-1} a_{2}, 2^{-1} a_{2}\right)  \tag{4.83}\\
& \frac{1}{N} \sum_{a_{1}, a_{2}=0} \tilde{O}\left(a_{1}, a_{2}\right)=O(0,0)
\end{align*}
$$

The Radon transform gives us the general marginal properties along any arbitrary axis of the $\mathbb{Z}(N) \times \mathbb{Z}(N)$ phase space:

$$
\begin{align*}
& \frac{1}{N} \sum_{a_{2}} O\left(a_{1} n+a-2 l, a_{1} m+a_{2} k\right)=\operatorname{Tr}\left[\hat{O} \hat{\Pi}_{p}^{(k, l)}(a-1)\right] \\
& \frac{1}{N} \sum_{a_{1}} O\left(a_{1} n+a-2 l, a_{1} m+a_{2} k\right)=\operatorname{Tr}\left[\hat{O} \hat{\Pi}_{q}^{(m, n)}(a-2)\right] \tag{4.84}
\end{align*}
$$

The inverse Radon transform is given by

$$
\begin{align*}
\tilde{O}\left(l a_{2}, k a_{2}\right) & =\sum_{a_{1}} \operatorname{Tr}\left[\hat{O} \hat{\Pi}_{p}^{(k, l)}\left(a_{1}\right)\right] \omega\left(-a_{1} a_{2}\right),  \tag{4.85}\\
\tilde{O}\left(n a_{2}, m a_{2}\right) & =\sum_{a_{2}} \operatorname{Tr}\left[\hat{O} \hat{\Pi}_{q}^{(m, n)}\left(a_{2}\right)\right] \omega\left(a_{1} a_{2}\right)
\end{align*}
$$

By changing the values of $(k, l)($ or $(m, n))$ and $a_{2}$ (or $a_{1}$ ), we can reach each of the $N^{2}$ points in phase space and determine the Weyl function from the generalized marginal probability. We know how to then find the Wigner function using the symplectic Fourier transform relationship.

## - Section 4.9

## Discrete *-product

It is geometrically interesting and calculationally useful to understand the phase space mapping of products of operators in Hilbert space. In the continuous case we have

$$
\begin{equation*}
\star \equiv e^{\frac{i \hbar}{2}\left(\overleftarrow{\partial}_{q} \vec{\partial}_{p}-\overleftarrow{\partial}_{p} \vec{\partial}_{q}\right)} \tag{4.86}
\end{equation*}
$$

A Fourier representation is the generalized convolution [9]

$$
\begin{align*}
\left(W_{\hat{A}} \star W_{\hat{B}}\right)(q, p)= & \frac{1}{\hbar^{2} \pi^{2}} \int d q^{\prime} d q^{\prime \prime} d p^{\prime} d p^{\prime \prime} W_{\hat{A}}\left(q^{\prime}, p^{\prime}\right) W_{\hat{B}}\left(q^{\prime \prime}, p^{\prime \prime}\right)  \tag{4.87}\\
& \times \exp \left(\frac{-2 i}{\hbar}\left(p\left(q^{\prime}-q^{\prime \prime}\right)+p^{\prime}\left(q^{\prime \prime}-q\right)+p^{\prime \prime}\left(q-q^{\prime}\right)\right)\right) \tag{4.88}
\end{align*}
$$

The expression that appears in the exponent multiplied by $-2 i / \hbar$ is twice the oriented area of a triangle $A$ in phase space with vertices at the respective primed pairs of coordinates. For $r=(q, p)$, we have

$$
\begin{equation*}
p\left(q^{\prime}-q^{\prime \prime}\right)+p^{\prime}\left(q^{\prime \prime}-q\right)+p^{\prime \prime}\left(q-q^{\prime}\right)=\left\langle r^{\prime \prime}, r^{\prime}\right\rangle+\left\langle r^{\prime}, r\right\rangle+\left\langle r, r^{\prime \prime}\right\rangle=2 \hat{A}\left(r, r^{\prime}, r^{\prime \prime}\right) \tag{4.89}
\end{equation*}
$$

The analogous equation in the discrete case shows up as follows. Let $(\hat{B} \hat{C})_{\alpha}$ denote the Wigner function for the operator equation to the operator product evaluated at point $\alpha$. We have

$$
\begin{equation*}
(\hat{B} \hat{C})_{\alpha}=\operatorname{Tr}[(\hat{B} \hat{C}) \hat{A}(\alpha)] \tag{4.90}
\end{equation*}
$$

We can expand $\hat{B}$ and $\hat{C}$ as their Wigner functions $B_{\beta}$ and $C_{\gamma}$ times the associated displaced parity operators:

$$
\begin{equation*}
\hat{B}=\frac{1}{N} \sum_{\beta} B_{\beta} \hat{A}_{\beta}, \quad \hat{C}=\frac{1}{N} \sum_{\gamma} C_{\gamma} \hat{A}_{\gamma} . \tag{4.91}
\end{equation*}
$$

Rearranging the resulting expression after plugging in the expansions, we find

$$
\begin{align*}
(\hat{B} \hat{C})_{\alpha} & =\frac{1}{N^{2}} \sum_{\beta \gamma} B_{\beta} C_{\gamma} \operatorname{Tr}\left[\hat{A}_{\alpha} \hat{A}_{\beta} \hat{A}_{\gamma}\right]  \tag{4.92}\\
& =\frac{1}{N} \sum_{\beta \gamma} B_{\beta} C_{\gamma} \Gamma_{\alpha \beta \gamma} \tag{4.93}
\end{align*}
$$

where $\Gamma_{\alpha \beta \gamma}=\frac{1}{N} \operatorname{Tr}\left[\hat{A}_{\alpha} \hat{A}_{\beta} \hat{A}_{\gamma}\right]$ is the 'three-point structure function' [143]. This equation defines the discrete $\star$-product:

$$
\begin{equation*}
(B \star C)_{\alpha}=\frac{1}{N} \sum_{\beta \gamma} B_{\beta} C_{\gamma} \Gamma_{\alpha \beta \gamma} . \tag{4.94}
\end{equation*}
$$

$\Gamma_{\alpha \beta \gamma}$ is an interesting object because it encodes the mathematical structure of the theory. It is a function of three points in phase space and for the odd prime dimensional case, can be written as

$$
\begin{equation*}
\Gamma_{\alpha \beta \gamma}=\frac{1}{N} \exp \left(\frac{4 \pi i}{N} \mathscr{A}_{\alpha \beta \gamma}\right) \tag{4.95}
\end{equation*}
$$

where $\mathscr{A}_{\alpha \beta \gamma}$ is the area of the triangle $(\alpha, \beta, \gamma)$ introduced in section 4.2. It has properties that can be simply proved graphically with triangles including the cyclical property $\Gamma_{\alpha \beta \gamma}=\Gamma_{\gamma \alpha \beta}=\Gamma_{\alpha \gamma \beta}^{*}$ and translational property $\Gamma_{\alpha+\delta, \beta+\delta, \gamma+\delta}=\Gamma_{\alpha, \beta, \gamma}$.

## Finite Fields and Quantum Nets

We have thoroughly discussed the nature of discrete phase space for the case of prime dimensions. Now we briefly comment on how this may be extended to the case of prime power dimensions via the use of finite fields. The physical significance of this approach is for a system of $n$ particles having a $p$-dimensional Hilbert space such that $p^{n}=N$.

Because of the properties of a field, there are no issues with labeling the coordinates of our horizontal and vertical axes with elements of a finite field $\mathbb{F}_{N}$. Our understanding of lines as sets of points satisfying some linear equation is still valid but now the arithmetic is done over elements of $\mathbb{F}_{N}$. As before, parallel lines will not intersect and nonparallel lines intersect at only one point.

Following Gibbons et al. [53], to each line in phase space, we assign a pure quantum state. The quantum net $\hat{Q}$ is defined such that for each line, $\lambda, \hat{Q}(\lambda)$ is the operator which projects onto the pure state associated with $\lambda$. The Wigner function representing our system with $N$ states should have the property that when summed over a line $\lambda$, the result is the probability that the quantum system is in the state $\hat{Q}(\lambda)$. If $\hat{\rho}$ is the density operator of the system, then

$$
\begin{equation*}
\sum_{\alpha \in \lambda} W_{\alpha}=\operatorname{Tr}[\hat{\rho} \hat{Q}(\lambda)] . \tag{4.96}
\end{equation*}
$$

This is the condition that determines the relationship between $\hat{\rho}$ and $W$ for a specific quantum net $\hat{Q}$.

When assigning states to lines, the operator $\hat{Q}(\lambda)$ must satisfy the property of translational covariance which we define below. A displacement operator $\hat{D}_{\alpha}$ associated with a point $\alpha=\left(\alpha_{1}, \alpha_{2}\right)$ must preserve the tensor product structure of the
composite system as it acts on each individual particle. Note that we now use $\alpha_{j}$ to denote the coordinates in the finite field for point $\alpha$. We expand $\alpha_{1}$ and $\alpha_{2}$ into a field basis decomposition in a certain way with expansion coefficients $a_{1} j$ and $a_{2} j$ such that

$$
\begin{equation*}
\alpha_{1}=\sum_{j}^{N-1} a_{1 j} w \tilde{e}_{j} \tag{4.97}
\end{equation*}
$$

and

$$
\begin{equation*}
\alpha_{2}=\sum_{j}^{N-1} a_{2 j} e_{j} \tag{4.98}
\end{equation*}
$$

where $w$ is any element of the finite field $\mathbb{F}_{N}$. The displacement operator for point $\alpha$ is

$$
\begin{equation*}
\hat{D}_{\alpha}=\otimes_{j=0}^{n-1} \hat{X}^{a_{2 j}} \hat{Z}^{a_{1 j}} \tag{4.99}
\end{equation*}
$$

The property of translational covariance is defined such that

$$
\begin{equation*}
\hat{Q}\left(\Delta_{\alpha} \lambda\right)=\hat{D}_{\alpha} \hat{Q}(\lambda) \hat{D}_{\alpha}^{\dagger} \tag{4.100}
\end{equation*}
$$

where $\Delta_{\alpha}$ just translated points in the phase space (not to be confused with the operator $\hat{D}_{\alpha}$ ). This will lead to a transformation by $\hat{D}_{\alpha}$ of a quantum state when the line $\lambda$ associated with this state is translated by a vector $\alpha$.

We can now recover a phase point operator by inverting Eq. (4.96) and expressing $W_{\alpha}$ in terms of $\hat{\rho}$. Remember that there are $N+1$ lines through the point $\alpha$ and that each point $\beta$ in phase space lies on one of these lines. The summation over all points lying on lines that include $\alpha, \sum_{\lambda \ni \alpha} \sum_{\beta \in \alpha}$, therefore includes $N+1$ counts of $\alpha$ and a single count of every other point. This leads to

$$
\begin{equation*}
W_{\alpha}=\frac{1}{N}\left[\left(\sum_{\lambda \ni \alpha} \sum_{\beta \in \alpha} W_{\beta}\right)-\sum_{\gamma} W_{\gamma}\right] \tag{4.101}
\end{equation*}
$$

Inserting Eq. (4.96) in for $W$, we have

$$
\begin{equation*}
W_{\alpha}=\frac{1}{N}\left[\sum_{\lambda \ni \alpha} \operatorname{Tr}[\hat{\rho} \hat{Q}(\lambda)]-1\right]=\frac{1}{N} \operatorname{Tr}\left(\hat{\rho} \hat{A}_{\alpha}\right) \tag{4.102}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{A}_{\alpha}=\left[\sum_{\lambda \ni \alpha} \hat{Q}(\lambda)\right]-\mathbb{1} . \tag{4.103}
\end{equation*}
$$

Directly from the definition, one can prove that $\hat{A}_{\alpha}$ has special properties:
(i) $\hat{A}_{\alpha}$ is Hermitian.
(ii) $\operatorname{Tr} \hat{A}_{\alpha}=1$.
(iii) $\operatorname{Tr} \hat{A}_{\alpha} \hat{A}_{\beta}=N \delta_{\alpha \beta}$.
(iv) $\sum_{\alpha \in \lambda} \hat{A}_{\alpha}=N Q(\lambda)$.
$\ulcorner$ Section 4.11

## Summary

We have achieved a very pleasing Wigner function. The phase point operators that define it form a unitary operator basis that reflects the symmetry of the underlying space. Namely, the phase point operators reflect the symplectic symmetry as an orbit under the symplectic group $S p[\mathbb{Z}(N)]$. Although areas must be preserved, there is no sense of distance as any two points can be transformed into any other two. This permutation symmetry (with two exceptions in dimensions two and eight) uniquely determines the discrete Wigner function [145].

## Chapter 5

## Discrete Phase Space Dynamics

In chapter 4, we leaned heavily on analogies with the structure of the continuous Wigner function to build a discrete version. We found some interesting group properties making the discrete Wigner function all the more utilitarian. We even had a brief glimpse of some mysterious connections to mutually unbiased bases via the geometric properties of the discrete phase space. As in the continuous case, we have become acquainted with the phase space ecosystem through the mapping from Hilbert space and the static properties of phase space. Now it is time to make the discrete Wigner functions dance.

Although there is a large literature devoted to studying quantum mechanics in discrete phase space, little attention has been paid to the nature of dynamical maps in this arena. Wootters set the framework for our study in his 1987 paper [143]. We have already argued that the mapping between Hilbert and discrete phase space that he introduced is unique in desirable ways. He also examines how discrete Wigner functions evolve unitarily in time.

We have mentioned some applications for quantum processes in quantum information [13, 94, 50, 29, 107, 90, 105, 93]. Cohendet et al. [27] formulated a Markovian jump process and noted that transition probabilities would be negative. The exis-
tence of negative transition probabilities is one of the results that we rediscovered and present in this chapter. Cohendet et al. use an interesting extended phase space which is a Cartesian product of the normal discrete phase phase with a binary variable $\sigma= \pm 1$. Their initial treatment was restricted to integer spin systems or $\operatorname{dim}(\mathcal{H})$ odd, but was generalized in a later work [28]. They give a description based on harmonic analysis on finite groups and use this to construct discrete Weyl operators (which play the role of our displacement operators) and Wigner functions. The interpretation they arrive at is of two classical Markov processes that are exchanged at random times. In another related work, Galetti and Ruzzi [49] analyze the discrete Wigner function dynamical equation and fashion a "discrete Liouvillian" that plays the same role as the transition rates that we will soon discuss. They go on to develop a time interval operator and comment on how the jumps between certain locations in phase space is determined by the time and Hamiltonian and not the initial state.

In this chapter, we study the dynamics of closed quantum mechanical systems whose number of Hilbert space dimensions is equal to any odd prime $N$. We will find the concrete example of a ring lattice with $N$ sites between which a particle can hop especially illustrative. However, one could also imagine a spin-one particle or some exotic higher spin boson model that retains our constraint of odd primes.

After introducing the discrete equivalent of the Moyal bracket for closed systems, we explore the construction of a Wigner current. The Wigner current will be factored into contributions from Wigner function and a transition rate. We will find that the transition rate holds more primary interest than the Wigner current itself. Investigation of the transition rates will enable characterization of all kinds of dynamics that may occur on the phase space for closed systems. Our understanding of classical probabilistic processes will be forced to be generalized as negativity enters the picture again; transition rates can take on negative values! We will derive a con-
straint on the set of transition rates and give a complete geometrical interpretation of this constraint in the space of transition rates. A set of rules will be laid out that determines when exactly a set of transition rates will be legal quantum dynamics on phase space. Along the way, the relationship between quantum dynamics and time can be interpreted in a new light.

- Section 5.1


## Discrete Wigner Current

Motivated by the fact that the Wigner current approach proved quite illuminating for our study of quantum dynamics in continuous phase space, we turn to the task of developing a similar method for finite dimensional quantum systems. We can immediately notice that there will be some significant differences between the continuous and the discrete cases. First of all, the discrete setting will obviously not have a continuous derivative so the analogue of the continuity equation will look quite different. Furthermore, continuous sets have an intrinsic sense of distance where currents take a particle from a starting point to a distant point along intermediary points. The discrete setting has no such sense of distance and currents can exist in principle between one point and any other point in the space, i.e. they can be non-local. Let us start with quantum dynamics in Hilbert space and watch these ideas take shape as we map into discrete phase space.

The von Neumann equation describes the closed system dynamics of density operators in Hilbert space:

$$
\begin{equation*}
\dot{\hat{\rho}}=\frac{1}{i \hbar}[\hat{H}, \hat{\rho}] . \tag{5.1}
\end{equation*}
$$

Expressing the operators in this equation in terms of the phase point operator basis introduced in the last chapter, we can derive the dynamical equation for Wigner
functions in discrete phase space. With $\hat{\rho}=\sum_{\alpha} W_{\alpha} \hat{A}_{\alpha}$ and $\hat{H}=\sum_{\beta} H_{\beta} \hat{A}_{\beta}$ we find

$$
\begin{equation*}
\frac{\partial W_{\alpha}}{\partial t}=\frac{1}{i \hbar} \sum_{\beta \gamma} \Gamma_{\alpha \beta \gamma}\left(H_{\beta} W_{\gamma}-W_{\beta} H_{\gamma}\right) \tag{5.2}
\end{equation*}
$$

where the three-point structure function $\Gamma_{\alpha \beta \gamma}=(1 / N) \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{A}_{\beta} \hat{A}_{\gamma}\right)$ plays an important role. Eq. (5.2) is just a representation of the discrete Moyal bracket

$$
\begin{align*}
\frac{\partial W}{\partial t} & =\frac{1}{i \hbar}\{H, W\}_{D M B}  \tag{5.3}\\
& =\frac{1}{i \hbar}(H \star W-W \star H) \tag{5.4}
\end{align*}
$$

with the star product defined in Eq. (4.94).
Using the identity $\Gamma_{\alpha \gamma \beta}=\Gamma_{\alpha \beta \gamma}^{*}$, we can rearrange this equation so that the summand is only proportional to the Wigner function evaluated at one point:

$$
\begin{equation*}
\frac{\partial W_{\alpha}}{\partial t}=\frac{1}{i \hbar} \sum_{\beta \gamma} W_{\gamma}\left[H_{\beta}\left(\Gamma_{\alpha \beta \gamma}-\Gamma_{\alpha \beta \gamma}^{*}\right)\right] \tag{5.5}
\end{equation*}
$$

To formulate a current in this space, we are looking for a quantity $\tilde{\mathscr{J}}_{\alpha \gamma}$ that 'carries' quasiprobability to one point $\alpha$ from another point $\gamma$. Any change with respect to time of the value of the Wigner function at some point $\alpha$ will be equal to the sum of the currents flowing from all points $\gamma$ into $\alpha$. We can therefore define the current intrinsically from Eq. (5.5) in conjunction with the requirement that:

$$
\begin{equation*}
\frac{\partial W_{\alpha}}{\partial t}=\sum_{\gamma} \tilde{\mathscr{J}}_{\alpha \gamma} \tag{5.6}
\end{equation*}
$$

We now have a candidate for the discrete Wigner current that is

$$
\begin{equation*}
\tilde{\mathscr{J}}_{\alpha \gamma} \equiv \frac{1}{i \hbar} W_{\gamma}\left[\sum_{\beta} H_{\beta}\left(\Gamma_{\alpha \beta \gamma}-\Gamma_{\alpha \beta \gamma}^{*}\right)\right] \tag{5.7}
\end{equation*}
$$

Note that our subscript labeling convention for dynamical objects that depend on two points places the point of arrival in the first subscript and the point of departure in the second. $\tilde{\mathscr{J}}_{\alpha \gamma}$ is the current from point $\gamma$ to point $\alpha$.

A quick calculation reveals the fact that this current is not antisymmetric as one would expect for most usual currents flowing in a continuous space:

$$
\begin{align*}
\tilde{\mathscr{J}}_{\gamma \alpha} & =\frac{1}{i \hbar} W_{\alpha}\left[\sum_{\beta} H_{\beta}\left(\Gamma_{\gamma \beta \alpha}-\Gamma_{\gamma \beta \alpha}^{*}\right)\right]  \tag{5.8}\\
& =\frac{i}{\hbar} W_{\alpha}\left[\sum_{\beta} H_{\beta}\left(\Gamma_{\alpha \beta \gamma}-\Gamma_{\alpha \beta \gamma}^{*}\right)\right] . \tag{5.9}
\end{align*}
$$

Therefore, $\tilde{\mathscr{J}}_{\alpha \gamma} \neq-\tilde{\mathscr{J}}_{\gamma \alpha}$ unless $W_{\alpha}=-W_{\gamma}$. It is then possible to think about the discrete Wigner currents as flowing along channels where for each pair of points, there exist two uni-directional channels, one flowing in each direction.

However, in this setting it is possible to antisymmetrize the currents by noting that inside the summations over $\gamma$ and $\beta$ which occur in Eq. (5.5), adding any function of position $\alpha$ (call it $F_{\alpha}$ ) to the factor $W_{\gamma} H_{\beta}$ that multiplies $\left(\Gamma_{\alpha \beta \gamma}-\Gamma_{\alpha \beta \gamma}^{*}\right)$ is effectively adding zero. It is straightforward to show that if $\alpha \neq \beta$,

$$
\begin{align*}
\sum_{\gamma} F_{\alpha} \Gamma_{\alpha \beta \gamma} & =\frac{1}{N} F_{\alpha} \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{A}_{\beta} \sum_{\gamma} \hat{A}_{\gamma}\right)  \tag{5.10}\\
& =\frac{1}{N} F_{\alpha} \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{A}_{\beta}\right)  \tag{5.11}\\
& =0 \tag{5.12}
\end{align*}
$$

We enforce antisymmetry of the discrete Wigner current by adding such a term that has a zero contribution and define our antisymmetrized discrete Wigner current:

$$
\begin{equation*}
\mathscr{J}_{\alpha \gamma} \equiv \frac{1}{i \hbar}\left(W_{\alpha}+W_{\gamma}\right)\left[\sum_{\beta} H_{\beta}\left(\Gamma_{\alpha \beta \gamma}-\Gamma_{\alpha \beta \gamma}^{*}\right)\right] . \tag{5.13}
\end{equation*}
$$

One interesting difference between $\tilde{\mathcal{J}}_{\alpha \gamma}$ and $\mathscr{J}_{\alpha \gamma}$ is that the sum of $\tilde{\mathscr{J}}_{\alpha \gamma}$ from a single point $\gamma$ to all other points $\alpha$ is zero, $\sum_{\alpha} \tilde{\mathscr{J}}_{\alpha \gamma}=0$; whereas the same sum for $\mathscr{J}_{\alpha \gamma}$ is just $\sum_{\alpha} \mathscr{J}_{\alpha \gamma}=-\frac{\partial W_{\alpha}}{\partial t}$. This emphasizes that antisymmetrized currents can thus be thought of as consisting of two unidirectional channels for each pair of phase-space points.

The form of $J_{\alpha \gamma}$ is curious due to the factor of $\left(W_{\alpha}+W_{\gamma}\right)$. The effect of this is to enable flow out of a point $\gamma$ even if there is no quasiprobability density at the point $\gamma$. This is a strange claim but is fitting with the possibility of having negative values for the Wigner function.

The motivation for investigating the antisymmetrization comes from the continuous realm where currents are described with continuous vector fields. If one looks at a fluid flowing in a pipe we just use a negative sign to indicate whether the fluid is leaving or arriving at a segment of the pipe. As an example, the amount flowing from segment 1 to segment 2 is just the negative of the amount flowing from segment 2 to segment 1. When dealing with discrete systems, we must think more carefully. For example, for simple classical probabilistic systems, the probability transition rates are never negative. The vectoral nature of flows in continuous space is quite different than flows between discrete states that only have a global constraint of conservation. Therefore, it is interesting to learn that we can antisymmetrize the Wigner current for finite dimensional quantum systems. However, this might not be so surprising because we already know that we are dealing with quasiprobabilities that can take negative values - we will draw a stranger conclusion shortly.

## - Section 5.2

## Wigner Transition Rates

We can decompose the discrete Wigner current into a product of the Wigner functions and the quantum phase space transition rate $\mathscr{J}_{\alpha \gamma}=\left(W_{\alpha}+W_{\gamma}\right) r_{\alpha \gamma}$ where

$$
\begin{equation*}
r_{\alpha \gamma}=\frac{1}{i \hbar}\left[\sum_{\beta} H_{\beta}\left(\Gamma_{\alpha \beta \gamma}-\Gamma_{\alpha \beta \gamma}^{*}\right)\right]=\frac{2}{\hbar} \sum_{\beta} \operatorname{Im}\left(\Gamma_{\alpha \beta \gamma}\right) H_{\beta} . \tag{5.14}
\end{equation*}
$$

The transition rate has units of frequency and intuitively gives the rate of transfer of quasiprobability to point $\alpha$ from point $\gamma$. We could call it transition probability per unit time if it were not for the issue that probabilities are always positive. Unlike the current, the transition rate is intrinsically antisymmetric, $r_{\alpha \gamma}=-r_{\gamma \alpha}$. If we had defined the transition rate using $\tilde{\mathscr{J}}=W_{\gamma} r_{\alpha \gamma}$, we would get the same resultant transition rate in Eq (5.14).

Because of the antisymmetry property, there will always be transition rates that are negative unless we have the trivial example where each transition rate from one point to another is zero. Just as the idea of negative probability was a shock to the system, this is a further affront to our preconceptions of how a probabilistic process must work! We have to accept that the quantum dynamical behavior in this space necessitates putting a minus sign in front of the quasiprobability that is being transferred for certain situations.

What is important to emphasize is that the transition rate $r_{\alpha \gamma}$ is in no way dependent on any Wigner function. This is a new negativity that appears in addition to the Wigner function negativity. $r_{\alpha \gamma}$ is only dependent on the Hamiltonian and the three-point structure function.

## - Section 5.3

## Example: Free Particle on a Ring

As an example, picture a simple ring with some prime number $N$ of lattice sites. A particle free to hop from site to site can be described by the Hamiltonian $\hat{H}=$ $\left(\hat{X}+\hat{X}^{\dagger}\right)$ where $\hat{X}$ is the shift operator defined in Eq. (4.19). We could have included an additional factor in the Hamiltonian but this will just speed up the dynamical processes. To derive this Hamiltonian, start with the definition of momentum given in Eq. (4.18). We make the usual claim that $\hat{H} \propto \hat{p}^{2}=\sum_{j=0}^{N-1} j^{2}|\hat{p} ; j\rangle\langle\hat{p} ; j|$. Applying the discrete Fourier transform to restate this equation in the position basis, we have $\hat{H} \propto \sum_{j=0}^{N-1}[|\hat{q} ; j+1\rangle\langle\hat{q} ; j|+|\hat{q} ; j\rangle\langle\hat{q} ; j+1|]$ which is what we claimed above.

Recall that $\operatorname{Tr} \hat{X}^{n} \hat{Z}^{m}=N \delta_{n, N} \delta_{m, N}$ from Eq. (4.34), $\omega^{n N}=1$ for $n \in \mathbb{N}$, and when we perform arithmetic modulo some prime $N$, we have $\frac{1}{2}=\frac{N+1}{2}$. Calculate $H_{\beta}$ as the expansion coefficients of phase point operators:

$$
\begin{align*}
H_{\beta} & =\frac{1}{N} \operatorname{Tr}\left[H \hat{A}_{b_{1}, b_{2}}\right]  \tag{5.15}\\
& =\frac{1}{N} \operatorname{Tr}\left[\left(\hat{X}+\hat{X}^{\dagger}\right) \frac{1}{N} \sum_{k, l} \omega^{b_{2} k-b_{1} l} \omega^{k l / 2} \hat{X}^{k} \hat{Z}^{l}\right] . \tag{5.16}
\end{align*}
$$

Performing the trace we have

$$
\begin{equation*}
H_{\beta}=\frac{1}{N^{2}} \sum_{k, l} \omega^{b_{2} k-b_{1} l} \omega^{k l / 2}\left(N \delta_{k, N-1} \delta_{l, N}+N \delta_{k, N+1} \delta_{l, N}\right) \tag{5.17}
\end{equation*}
$$

and the summations leave

$$
\begin{equation*}
H_{\beta}=\frac{1}{N}\left(\omega^{b_{2}(N-1)}+\omega^{b_{2}(N+1)}\right) . \tag{5.18}
\end{equation*}
$$

The form of the 3-point structure function is the same as Wootters' original con-
struction [143]:

$$
\begin{align*}
\operatorname{Im}\left(\Gamma_{\alpha \beta \gamma}\right) & =\frac{1}{N} \sin \left[\frac{4 \pi}{N}\left(a_{1} b_{2}+b_{1} c_{2}+c_{1} a_{2}-a_{2} b_{1}-b_{2} c_{1}-c_{2} a_{1}\right)\right]  \tag{5.19}\\
& =\frac{1}{2 i N}\left[\omega^{2\left(a_{1} b_{2}+b_{1} c_{2}+c_{1} a_{2}-a_{2} b_{1}-b_{2} c_{1}-c_{2} a_{1}\right)}-\text { c.c. }\right] .
\end{align*}
$$

Transition rate can now be calculated using the fact that the summations over roots of unity will collapse to zero unless the exponent is equal to zero in which case we have a factor of $N$ :

$$
\begin{align*}
r_{\alpha \gamma} & =\frac{2}{\hbar} \sum_{\beta} H_{\beta} \operatorname{Im}\left(\Gamma_{\alpha \beta \gamma}\right) \\
& =\frac{2}{\hbar} \frac{1}{2 i N} \sum_{b_{2}}\left(\omega^{b_{2}(N-1)}+\omega^{b_{2}(N+1)}\right)\left[\omega^{2\left(a_{1} b_{2}+c_{1} a_{2}-b_{2} c_{1}-c_{2} a_{1}\right)}-c . c .\right] \delta_{a_{2}, c_{2}}  \tag{5.20}\\
& =\frac{2}{\hbar} \sin \left(\frac{2 \pi}{N} c_{2}\right)\left[\delta_{a_{2}, c_{2}} \delta_{a_{1}-c_{1},-\frac{N+1}{2}}-\delta_{a_{2}, c_{2}} \delta_{a_{1}-c_{1},-\frac{N-1}{2}}\right] .
\end{align*}
$$

As indicated by the Kronecker deltas, this calculation shows that the current can only flow to point $\alpha$ from two points $\gamma$ which all have the same momentum coordinates and the position coordinates for the two $\gamma$ 's must be $\gamma_{1}=\alpha_{1}+\frac{N \pm 1}{2}$. Notice how this arises from the multiplier $(N \pm 1)$ of $b_{2}$ in the exponent of $\omega$. The sum over $b_{2}$ collapses the exponent down the very Kronecker delta we are discussing. This relationship defines the points that we will refer to as "across" from $\alpha_{1}$, picturing a lattice with $N$ sites physically embedded in Euclidean space. We will still use this terminology when we discuss more abstract examples with general Hamiltonians because when dealing with integers modulo some prime, there will always exist this idea of a lattice ring in for some observable that we may call position. Fig. (5.1) displays four arbitrary time snapshots in the evolution of a free particle initialized in an eigenstate of the position in the four leftmost images. The two rightmost images indicate how we may picture the transition rates behaving on the ring of discrete states embedded in Euclidean
space. We could have also plotted rings for momentum but this would have been quite uninteresting as there is no transition rate between momentum states.

## Free Particle Dynamics




$|0\rangle$




$$
\mathrm{t}=1 \text { (a.u.) }
$$

$$
|1\rangle
$$

|0〉
$|4\rangle$

Figure 5.1: Images of discrete Wigner functions at different time snapshots and the transition rates that generate the dynamics. We can see in the Wigner plots that the initial flow is across the lattice but does not change the marginal probabilities. It appears that second order in time processes transfer quasiprobability to the nearest neighbor sites of $|2\rangle$ which build up and positive marginal probability values.

Leaving behind the concrete example of a particle on a ring, imagine that we have some system with a Hamiltonian $\hat{H}=\left(\hat{Z}+\hat{Z}^{\dagger}\right)$. We don't need to go through the calculation for $r_{\alpha \gamma}$ because we can just recognize the symmetry with the example that we just studied. Roles of position and momentum will just be flipped. Our resulting transition rates will be the same as the last but with the subscripts 1 and 2 flipped. The current will flow only between points with the same position coordinates but with
momentum coordinate differing by $\frac{N \pm 1}{2}$. Again, we have flow across a ring lattice but now the ring has sites determined by momenta.

Both of these examples show us something surprising. If one knew that $\hat{X}$ and $\hat{Z}$ respectively displace position and momentum states by a single step, it seems intuitive that the currents associated with the Hamiltonians above might flow to the nearest neighbor locations. There is something deeper going on.

Thinking back to the free particle on the lattice, let us imagine a particle initially localized at some position labelled $|x ; 2\rangle$. We will study the transition rates between positions at an arbitrary nonzero momentum value $p$. The top-right image in Fig. (5.1) shows up to a sign how the initial transfer of quasiprobability occurs in one time step. The initial value at site $(2, p)$ is unchanged due to the fact that equal amounts of negative and positive quasiprobability are transferred to sites $\left(\frac{N+1}{2}, p\right)$ and $\left(\frac{N-1}{2}, p\right)$.

The bottom-right image in Fig. (5.1) shows what occurs in the next time step. It is again described by the elemental flows across the circle but now there are three sites that can initiate. We know what happens from site $(0, p)$. The current from sites $\left(\frac{N+1}{2}, p\right)$ and $\left(\frac{N-1}{2}, p\right)$ result in a cumulative negative contribution to site $(0, p)$ and each have their own positive contribution to sites $( \pm 1, p)$. This will hold true for all values of $p$ momentum. It is this second step in time that changes the marginal probabilities, reducing the probability at the initial position $|2\rangle$ and increasing it at its nearest neighbors $|1\rangle$ and $|3\rangle$.

We have been ignoring the sinusoidal dependence but this also has an interesting role to play. The flows across the circle could be construed as nonlocal effects if we again imagine the situation of the ring of position states actually representing a ring with positions embedded in space. However, it is the sinusoidal nature of the magnitude of the currents that this nonlocal behavior is not observed in the marginals. The cumulative current out of a certain position column with nonzero (but always
positive) marginal probability will never effect a change in marginal probability in one time step. It takes the second time step for the marginal probability to decrease at its point of origin and increase someplace else. The quantum Zeno effect has its origins in this property of quantum mechanics [95].

We have discovered in phase space a general property of quantum mechanics. Probabilities do not change in a single time step, they require two. This can be understood in the Hilbert space context of unitary dynamics. Wavefunctions evolve as unitary rotations so it is probability amplitudes that change in a single time step. Probabilities are equal to the modulus squared of the amplitudes.

## - Section 5.4

## Visualizing Transition Rates

Part of the joy and utility of employing phase space methods comes from the information that can be obtained from visualizations. The transition rates are no exception in this regard. It is clear that transition rates live in a much larger space that functions on phase space because they are indexed by two points $\alpha$ and $\gamma$. One way to visualize them is on a product space of two phase spaces. The space has $N^{2}$ horizontal coordinates labeled with a base- $N$ counting system. The first number corresponds to the position $a_{1}$ of phase space point $\alpha$ and the second number corresponds to the position $c_{1}$ of the phase space point $\gamma$. Likewise, there are $N^{2}$ vertical coordinates labeled in the same manner but with momenta of points $\alpha$ and $\gamma$ as opposed to positions. A major grid line marks transitions from one coordinate belonging to $\alpha$ to the next. The minor grid line plays the same roll for points $\gamma$.

The trick to reading these plots is to gain proficiency zooming in and out between the major grid lines. If you want to understand the flow into some point $\alpha$, find the correct square inside the major grid lines and treat this as a phase space of its own.

Now, identify the corresponding square for $\alpha$ in the $\gamma$-space; this is the point to which all the other points flow with rates indicated by the heat map.

Transition rates from $\left(c_{1}, c_{2}\right)$ to $\left(a_{1}, a_{2}\right)$ for $\hat{H}=X+X^{\dagger}$


Figure 5.2: Transition rates from points $\left(c_{1}, c_{2}\right)$ to points $\left(a_{1}, a_{2}\right)$ are plotted with labels $\left(a_{1} c_{1}, a_{2} c_{2}\right)$ for a 5 -dimensional Hilbert space. A colorbar indicates the magnitude of the individual transition rates. Notice that for each major grid box, only two points have flow to the appropriate point of arrival.

As an example, consider the case of the free particle on a ring with $N=5$ positional sites. Fig. (5.2) displays the information contained in the set of transitions rates equivalent to this Hamiltonian. Suppose we are interested in learning about the flow of quasiprobability into the point $(0,4)$. We must narrow our focus to the major grid box labelled with the first position coordinate $a_{1}=0$ and the first momentum coordinate $a_{2}=4$. This is the upper left of the twenty-five major grid boxes. Once focused on the appropriate major grid box, identify the minor grid box that has the
corresponding position in the major grid box, as the major grid box has in the whole plot; it is labelled $(00,44)$. Imagining the major grid box as one phase space, the point we have just identified is the point to which we plot the quasiprobability flow from all other points. A trick to identifying the point to which the flow occurs, is to realize that the pairs of position and momentum coordinates that label it always occur as repeated values, such as $(33,11)$. The magnitude of individual transition rates is indicated by the color and the corresponding numerical value can be read off from the accompanying color bar. Differences among these are not particularly easy to discern in Fig. (5.2) but it will become more interesting as there exist more nonzero points to plot. What we can see in this figure is the flow 'across' between points with position $q$ and points with position $q+(N \pm 1) / 2$. There is no transition between states of differing momenta.

Once we have familiarized ourselves with the process of reading these plots, we can look at larger dimensional spaces. In this case, it is impractical to retain all the labels for the individual boxes representing a pair of points. Unless we are truly interested in the behavior at one box representing a pair of phase space points, we will do away with the labels and instead focus on global properties. Let us again focus on the case of the free particle but move to $N=11$ dimensions as in Fig. (5.3). Focus on a single column and note how the magnitude of the transition rates change in a sinusoidal manner, covering one period over the column. Looking at an individual row, we can see that the magnitudes stay constant throughout. As we change from one major grid box to the next, the pair of nonzero transition rates is displaced by the corresponding amount in the minor grid. Again, this is just a reflection of the flow only being possible between points with position $q$ and points with position $q+(N \pm 1) / 2$. The bottom row shows no flow where the momentum is zero which aligns with what we would expect for the case of a free particle.

Transition rates from $\left(c_{1}, c_{2}\right)$ to $\left(a_{1}, a_{2}\right)$ for $\hat{H}=X+X^{\dagger}$


Figure 5.3: Transition rates for $\hat{H}=\hat{X}+\hat{X}^{\dagger}$ with an 11-dimensional Hilbert space.

Fig. (5.4) displays transition rates for $\hat{H}=\hat{Z}+\hat{Z}^{\dagger}$ and we see a symmetry with those of the free particle. This is just an extension of the symmetry the Schwinger noted in the clock and shift operators. Fig. (5.5) exhibits how additive combinations of Hermitian displacement terms (a Harper-like Hamiltonian [92]) results simply in an additive combination of their respective transition rates in the rate space. We can make our lives easier by expanding any Hamiltonian over the basis of displacement operators and examining it term by term. We will see that this expansion makes it quite straightforward to decode and characterize the complete dynamics in phase space. As we study more general Hamiltonians, the transition rates will exhibit many

Transition rates from $\left(c_{1}, c_{2}\right)$ to $\left(a_{1}, a_{2}\right)$ for $\hat{H}=Z+Z^{\dagger}$


Figure 5.4: Transition rates for $\hat{H}=\hat{Z}+\hat{Z}^{\dagger}$ show a symmetry with the those for $\hat{H}=\hat{X}+\hat{X}^{\dagger}$.
similar patterns to the ones we have seen. We will be able to search for lines in the major grid of transition rate space that have constant rate magnitudes, and also understand the sinusoidal nature.

## - Section 5.5 <br> General Transition Rates

We will now broaden our understanding of transition rates by studying Hamiltonians with higher order terms in $\hat{X}$ and $\hat{Z}$. Take our general Hamiltonian to be $H=$

Transition rates from $\left(c_{1}, c_{2}\right)$ to $\left(a_{1}, a_{2}\right)$ for $\hat{H}=X+X^{\dagger}+Z+Z^{\dagger}$


Figure 5.5: Transition rates for the Harper-like Hamiltonian $\hat{H}=\hat{X}+\hat{X}^{\dagger}+\hat{Z}+\hat{Z}^{\dagger}$ show how two Hermitian terms can be analyzed separately and superposed.
$\sum_{n, m=0}^{N} \kappa_{n m}\left[\hat{X}^{n} \hat{Z}^{m}+\left(\hat{X}^{n} \hat{Z}^{m}\right)^{\dagger}\right]$ where the scalar function $\kappa(n, m)$ gives the strength of one particular term in the Hamiltonian. We will proceed to ignore $\kappa(n, m)$ as this just scales the transition rates of the separate terms. As we have just mentioned, the contribution of each term simply adds linearly so we will only be interested in one general term of the Hamiltonian. For the phase space representation of such a term,
we find

$$
\begin{align*}
\left(\hat{X}^{n} \hat{Z}^{m}+\hat{Z}^{\dagger m} \hat{X}^{\dagger n}\right)_{\beta}= & \frac{1}{N} \operatorname{Tr}\left[\left(\hat{X}^{n} \hat{Z}^{m}+\hat{Z}^{\dagger m} \hat{X}^{\dagger n}\right) \hat{A}_{\beta}\right]  \tag{5.21}\\
= & \frac{1}{N}\left(\omega^{b_{2}(N-n)-b_{1}(N-m)} \omega^{-n m / 2}\right.  \tag{5.22}\\
& \left.+\omega^{b_{2}(N+n)-b_{1}(N+m)} \omega^{n m / 2}\right) . \tag{5.23}
\end{align*}
$$

In the free particle case, the multiplier $(N \pm 1)$ of $b_{2}$ in the exponent of $\omega$ played an important role. Now we have multipliers of $(N \pm n)$ and $(N \pm m)$ for $b 2$ and $b 1$, respectively. These multipliers end up determining the possible leaps in position and momentum and show up as a Kronecker delta $\delta_{a_{2}-c_{2}, N \pm n / 2}$. From the form of the Kronecker delta, it is clear that it takes 2 steps to reach a distance $\pm n$.

Now we continue with a calculation of the transition rate for our arbitrary Hamiltonian term.

$$
\begin{align*}
r_{\alpha \gamma}= & \frac{2}{\hbar} \sum_{\beta}\left(\hat{X}^{n} \hat{Z}^{m}+\hat{Z}^{\dagger m} \hat{X}^{\dagger n}\right)_{\beta} \operatorname{Im}\left(\Gamma_{\alpha \beta \gamma}\right)  \tag{5.24}\\
= & \frac{2}{\hbar} \frac{1}{2 i N^{2}} \sum_{b_{1}, b_{2}}\left(\omega^{b_{2}(N-n)-b_{1}(N-m)} \omega^{-n m / 2}+\omega^{b_{2}(N+n)-b_{1}(N+m)} \omega^{n m / 2}\right)  \tag{5.25}\\
& \times\left[\omega^{2\left(a_{1} b_{2}+b_{1} c_{2}+c_{1} a_{2}-a_{2} b_{1}-b_{2} c_{1}-c_{2} a_{1}\right)}-c . c .\right] \tag{5.26}
\end{align*}
$$

The summation over $b_{1}$ (or $b_{2}$ ) will have a null result unless $N-m+2 c_{2}-2 a_{2}=0$ (or $N-n+2 a_{1}-2 c_{1}=0$ ). This is equivalent to the Kronecker delta $\delta_{a_{2}-c_{2},(N-m) / 2}$
(or $\delta_{a_{1}-c_{1},-(N-n) / 2}$ ). Proceeding we have

$$
\begin{align*}
& r_{\alpha \gamma}=\frac{2}{2 i \hbar} {\left[\omega^{2\left(c_{1} a_{2}-c_{2} a_{1}\right)}-c . c .\right] }  \tag{5.27}\\
& \times\left(\delta_{a_{1}-c_{1},-(N-n) / 2} \delta_{a_{2}-c_{2},(N-m) / 2} \omega^{-n m / 2}\right.  \tag{5.28}\\
&\left.+\delta_{a_{1}-c_{1},-(N+n) / 2} \delta_{a_{2}-c_{2},(N+m) / 2} \omega^{n m / 2}\right)  \tag{5.29}\\
&=\frac{2}{\hbar} \sin \left[\frac{4 \pi}{N}\left(c_{1} a_{2}-c_{2} a_{1}\right)\right]  \tag{5.30}\\
& \times\left(\delta_{a_{1}-c_{1},-(N-n) / 2} \delta_{a_{2}-c_{2},(N-m) / 2} \omega^{-n m / 2}\right.  \tag{5.31}\\
&\left.+\delta_{a_{1}-c_{1},-(N+n) / 2} \delta_{a_{2}-c_{2},(N+m) / 2} \omega^{n m / 2}\right) . \tag{5.32}
\end{align*}
$$

The result is that it is possible to have a current for any position and momentum displacement in phase space. One just needs to have the power of Z or X be twice as large as the number of position or momentum steps you wish to displace.

Building off what we have just learned, if we want to have quasiprobability transition directly between nearest neighbor sites on a ring in Euclidean space, we will need up to a constant the Hamiltonian $\hat{H}=\hat{X}^{2}+\hat{X}^{\dagger}$. Figure (5.6) gives us a visualization that aligns with this notion. We can notice how the horizontal lines all have the same magnitude for nonzero rates. Additionally, the nonzero values in a column can be seen to cover two periods of a sine function. It is not a coincidence that this number two is the same as the power of $\hat{X}$. All of these properties will be derived in this chapter. A more complicated example with transition rates corresponding to the Hamiltonian $\hat{H}=\hat{X}^{7} \hat{Z}^{4}+\left(\hat{X}^{7} \hat{Z}^{4}\right)^{\dagger}$ is displayed in Fig. (5.7). A fun exercise is to search for the lines of constant magnitude and work to discern a sinusoidal relationship in the nonzero magnitudes of the rows and columns.

We now have characterized the transition rates for any possible Hamiltonian. Our calculations were facilitated by an expansion of the Hamiltonian over the basis of displacement operators. It might become complicated with many terms, we can


Figure 5.6: Transition rates for $\hat{H}=\hat{X}^{2}+\hat{X}^{\dagger}$ showing nearest neighbor interactions. A barplot of the sine function sampled eleven times evenly over two periods provides a visual aid to understand the sinusoidal nature of the nonzero values in various columns such as the second minor grid column from the left.
easily understand them separately on the space of transition rates.

## - Section 5.6

## Transition Rate Properties

We have now had some practice starting with a Hamiltonian and calculating the associated transition rates. However, we can adopt the viewpoint that we don't need a Hamiltonian to determine the dynamics. Instead, we can just start with a set of transition rates between each point in phase space. A Hamiltonian for an N dimensional system has $N^{2}-1$ free parameters. However, there exist $N^{2}\left(N^{2}-1\right)$

Transition rates from $\left(c_{1}, c_{2}\right)$ to $\left(a_{1}, a_{2}\right)$ for $\hat{H}=X^{7} Z^{4}+\left(X^{7} Z^{4}\right)^{\dagger}$


Figure 5.7: Transition rates for $\hat{H}=\hat{X}^{7} \hat{Z}^{4}+\left(\hat{X}^{7} \hat{Z}^{4}\right)^{\dagger}$ present a more complicated relational structure amongst the transition rates.
distinct pairs of points in phase space where transition may occur. This brings up some new questions. Is any set of transition rates that we can think of a legitimate candidate? This seems rather unlikely at first blush just due to the fact that the marginal probabilities must remain positive. Therefore, what conditions determine a legitimate set of transition rates? We explore this question in what follows.

### 5.6.1. Transition rates determine Hamiltonians

If we did have some way of creating sets of legitimate transition rates without consulting Hilbert state constructions, we might be interested in finding the corresponding

Hamiltonian.
We have expressed the transition rates $r_{\alpha \gamma}$ in terms of the Hamiltonian. The goal of this subsection is to invert this relation and write the Hamiltonian in terms of the transition rates. This can be done only up to an additive constant in the Hamiltonian, but note that such a constant has no effect on the dynamics.

Start with the definition of $r_{\alpha \gamma}$ :

$$
\begin{equation*}
r_{\alpha \gamma}=\frac{2}{\hbar} \sum_{\beta} \operatorname{Im}\left(\Gamma_{\alpha \beta \gamma}\right) H_{\beta}=\frac{1}{i \hbar} \sum_{\beta}\left(\Gamma_{\alpha \beta \gamma}-\Gamma_{\gamma \beta \alpha}\right) H_{\beta} . \tag{5.33}
\end{equation*}
$$

Again, $H_{\beta}$ is the phase space function representing the Hamiltonian, so that $\hat{H}=$ $\sum_{\beta} H_{\beta} \hat{A}_{\beta}$. Recall also the definition of $\Gamma_{\alpha \beta \gamma}$ :

$$
\begin{equation*}
\Gamma_{\alpha \beta \gamma}=\frac{1}{N} \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{A}_{\beta} \hat{A}_{\gamma}\right) . \tag{5.34}
\end{equation*}
$$

Inserting this definition into Eq. (5.33) and performing the sum over $\beta$, we get

$$
\begin{equation*}
r_{\alpha \gamma}=\frac{1}{N i \hbar}\left[\operatorname{Tr}\left(\hat{A}_{\alpha} \hat{H} \hat{A}_{\gamma}\right)-\operatorname{Tr}\left(\hat{A}_{\gamma} \hat{H} \hat{A}_{\alpha}\right)\right] . \tag{5.35}
\end{equation*}
$$

Now we use the fact that for any $N \times N$ matrix $\hat{M}, \sum_{\gamma} \operatorname{tr}\left(M A_{\gamma}\right) \hat{A}_{\gamma}=N M$. Multiply Eq. (5.35) by $\hat{A}_{\gamma}$ and sum over $\gamma$ :

$$
\begin{equation*}
\sum_{\gamma} r_{\alpha \gamma} \hat{A}_{\gamma}=\frac{1}{i \hbar}\left(\hat{A}_{\alpha} \hat{H}-\hat{H} \hat{A}_{\alpha}\right) \tag{5.36}
\end{equation*}
$$

Now multiply on the left by $\hat{A}_{\alpha}$ and sum over $\alpha$ :

$$
\begin{equation*}
\sum_{\alpha \gamma} r_{\alpha \gamma} \hat{A}_{\alpha} \hat{A}_{\gamma}=\frac{1}{i \hbar} \sum_{\alpha}\left(\hat{A}_{\alpha}^{2} \hat{H}-\hat{A}_{\alpha} \hat{H} \hat{A}_{\alpha}\right) \tag{5.37}
\end{equation*}
$$

From the definition of the $\hat{A}_{\alpha}$ 's, one can derive directly the following two facts:

$$
\begin{align*}
& \hat{A}_{\alpha}^{2}=\mathbb{1}  \tag{5.38}\\
& \sum_{\alpha} \hat{A}_{\alpha} \hat{M} \hat{A}_{\alpha}=N(\operatorname{Tr} \hat{M}) \mathbb{1} \tag{5.39}
\end{align*}
$$

where $\hat{M}$ is some $N \times N$ matrix. We can therefore simplify the right-hand side of Eq. (5.37) and write

$$
\begin{equation*}
\sum_{\alpha \gamma} r_{\alpha \gamma} \hat{A}_{\alpha} \hat{A}_{\gamma}=\frac{1}{i \hbar}\left[N^{2} \hat{H}-N(\operatorname{Tr} \hat{H}) \mathbb{1}\right] \tag{5.40}
\end{equation*}
$$

Now multiply by $i \hbar / N^{2}$ :

$$
\begin{equation*}
\frac{i \hbar}{N^{2}} \sum_{\alpha \gamma} r_{\alpha \gamma} \hat{A}_{\alpha} \hat{A}_{\gamma}=\hat{H}-(\operatorname{Tr} \hat{H}) \frac{1}{N} \mathbb{1} \tag{5.41}
\end{equation*}
$$

The right-hand side is a traceless version of the Hamiltonian $\hat{H}$. In this way the $r$ 's determine $\hat{H}$ up to an additive constant.

We can go a step further and write the phase space function $H_{\beta}$ (up to an additive constant) in terms of the $r$ 's. Multiply Eq. (5.41) on the left by $\hat{A}_{\beta}$ and take the trace. This gives us

$$
\begin{equation*}
\frac{i \hbar}{N} \sum_{\alpha \gamma} r_{\alpha \gamma} \Gamma_{\beta \alpha \gamma}=N H_{\beta}-\frac{1}{N} \operatorname{Tr} \hat{H} \tag{5.42}
\end{equation*}
$$

But $\operatorname{Tr} \hat{H}$ is the same as $\sum_{\delta} H_{\delta}$. So we have (using a familiar symmetry of $\Gamma$ )

$$
\begin{equation*}
\frac{i \hbar}{N^{2}} \sum_{\alpha \gamma} r_{\alpha \gamma} \Gamma_{\alpha \beta \gamma}^{*}=\frac{i \hbar}{N^{2}} \sum_{\alpha \gamma} r_{\alpha \gamma} \Gamma_{\beta \alpha \gamma}=H_{\beta}-\frac{1}{N^{2}} \sum_{\delta} H_{\delta} \tag{5.43}
\end{equation*}
$$

From the antisymmetry of $r_{\alpha \gamma}$ we also have

$$
\begin{equation*}
-\frac{i \hbar}{N^{2}} \sum_{\alpha \gamma} r_{\alpha \gamma} \Gamma_{\alpha \beta \gamma}=H_{\beta}-\frac{1}{N^{2}} \sum_{\delta} H_{\delta} . \tag{5.44}
\end{equation*}
$$

Adding Eqs. (5.43) and (5.44) and dividing by 2, we get

$$
\begin{equation*}
\frac{\hbar}{N^{2}} \sum_{\alpha \gamma} r_{\alpha \gamma} \operatorname{Im}\left(\Gamma_{\alpha \beta \gamma}\right)=H_{\beta}-\frac{1}{N^{2}} \sum_{\delta} H_{\delta} \tag{5.45}
\end{equation*}
$$

It's interesting that the imaginary part of $\Gamma_{\alpha \beta \gamma}$ is the only part that matters in Eqs. (5.33) and (5.45).

### 5.6.2. Generalization to other quantum nets

In the above derivation, Eq. (5.38) is specific to the case of odd prime $N$ for the Wigner function of Wootters (1987). But Eq. (5.38) is not actually necessary to get to Eq. (5.40); one needs only Eq. (5.39). We show here that Eq. (5.39) is valid for any quantum net. It follows that the conclusions of the above argument, that is, Eqs. (5.41) and (5.45), are valid for any quantum net.

Start with the following general expression for $\hat{A}_{\alpha}$ :

$$
\begin{equation*}
\hat{A}_{\alpha}=\left(\sum_{\lambda \ni \alpha} \hat{P}_{\lambda}\right)-\mathbb{1} \tag{5.46}
\end{equation*}
$$

where $\hat{P}_{\lambda}$ is the projection operator onto the quantum state assigned to the line $\lambda$. We have, then,

$$
\begin{equation*}
\sum_{\alpha} \hat{A}_{\alpha} \hat{M} \hat{A}_{\alpha}=\sum_{\alpha}\left[\left(\sum_{\lambda \ni \alpha} \hat{P}_{\lambda}\right)-\mathbb{1}\right] \hat{M}\left[\left(\sum_{\lambda^{\prime} \ni \alpha} \hat{P}_{\lambda^{\prime}}\right)-\mathbb{1}\right] . \tag{5.47}
\end{equation*}
$$

Consider first the sum $\sum_{\alpha} \sum_{\lambda \ni \alpha} \hat{P}_{\lambda}$. In this sum, every line appears $N$ times. Moreover, the $\hat{P}_{\lambda}$ 's associated with the lines of a given striation sum to the identity. There
are $N+1$ striations, so we have

$$
\begin{equation*}
\sum_{\alpha} \sum_{\lambda \ni \alpha} \hat{P}_{\lambda}=N(N+1) \mathbb{1} . \tag{5.48}
\end{equation*}
$$

Thus Eq. (5.47) can be rewritten as

$$
\begin{equation*}
\sum_{\alpha} \hat{A}_{\alpha} \hat{M} \hat{A}_{\alpha}=N^{2} \hat{M}-2 N(N+1) \hat{M}+\sum_{\alpha} \sum_{\lambda \ni \alpha} \sum_{\lambda^{\prime} \ni \alpha} \hat{P}_{\lambda} \hat{M} \hat{P}_{\lambda^{\prime}} \tag{5.49}
\end{equation*}
$$

In the triple sum, we are in effect summing over all pairs of lines that intersect, with a factor of $N$ if the two lines $\lambda$ and $\lambda^{\prime}$ are identical (since in that case there are $N$ values of $\alpha$ for which that "pair" of lines appears). We can write

$$
\begin{equation*}
\sum_{\alpha} \sum_{\lambda \ni \alpha} \sum_{\lambda^{\prime} \ni \alpha} \hat{P}_{\lambda} \hat{M} \hat{P}_{\lambda^{\prime}}=\sum_{\lambda, \lambda^{\prime}}\left(v_{\lambda, \lambda^{\prime}}+N \delta_{\lambda, \lambda^{\prime}}\right) \hat{P}_{\lambda} \hat{M} \hat{P}_{\lambda^{\prime}} \tag{5.50}
\end{equation*}
$$

where $\delta_{\lambda, \lambda^{\prime}}$ is the usual Kronecker delta, and $v_{\lambda, \lambda^{\prime}}$ has the value 1 if $\lambda$ and $\lambda^{\prime}$ intersect in exactly one point and is zero otherwise. Now, the part of the sum that $v_{\lambda, \lambda^{\prime}}$ is in can be broken up into sums over striations: there are $N+1$ striations in which $\lambda$ can appear, and for each of these, there are $N$ striations in which $\lambda^{\prime}$ can appear. Part of the sum has the value $N(N+1) \hat{M}$. (Again, the sum of the $\hat{P}_{\lambda}$ 's over any striation is the identity.) This leaves us with

$$
\begin{align*}
\sum_{\alpha} \hat{A}_{\alpha} \hat{M} \hat{A}_{\alpha} & =N^{2} \hat{M}-2 N(N+1) \hat{M}+N(N+1) \hat{M}+N \sum_{\lambda} \hat{P}_{\lambda} \hat{M} \hat{P}_{\lambda}  \tag{5.51}\\
& =-N M+N \sum_{\lambda} \hat{P}_{\lambda} \hat{M} \hat{P}_{\lambda}
\end{align*}
$$

We now have to find the value of $\sum_{\lambda} \hat{P}_{\lambda} \hat{M} \hat{P}_{\lambda}$, which is a sum over the one-dimensional projection operators representing a complete set of mutually unbiased bases. It helps to write the matrix $\hat{M}$ as a linear combination of the $\hat{P}_{\lambda}$ 's, which we can certainly
do. (We can write any matrix as a linear combination of the $\hat{A}_{\alpha}$ 's, and each $\hat{A}_{\alpha}$ is a linear combination of $\hat{P}_{\lambda}$ 's.) Write $\hat{M}=\sum_{\nu} m_{\nu} \hat{P}_{\nu}$. Then we can find $\sum_{\lambda} \hat{P}_{\lambda} \hat{M} \hat{P}_{\lambda}$ once we have found $\sum_{\lambda} \hat{P}_{\lambda} \hat{P}_{\nu} \hat{P}_{\lambda}$ :

$$
\begin{equation*}
\sum_{\lambda} \hat{P}_{\lambda} \hat{P}_{\nu} \hat{P}_{\lambda}=\hat{P}_{\nu} \hat{P}_{\nu} \hat{P}_{\nu}+\sum_{\lambda} v_{\lambda, \nu} \hat{P}_{\lambda} \hat{P}_{\nu} \hat{P}_{\lambda}=\hat{P}_{\nu}+\frac{1}{N} \sum_{\lambda} v_{\lambda, \nu} \hat{P}_{\lambda}=\hat{P}_{\nu}+\mathbb{1} . \tag{5.52}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
\sum_{\lambda} \hat{P}_{\lambda} \hat{M} \hat{P}_{\lambda}=\sum_{\nu} m_{\nu}\left(\hat{P}_{\nu}+\mathbb{1}\right)=\hat{M}+(\operatorname{tr} \hat{M}) \mathbb{1} . \tag{5.53}
\end{equation*}
$$

Putting this result back into Eq. (5.51), we finally get

$$
\begin{equation*}
\sum_{\alpha} \hat{A}_{\alpha} \hat{M} \hat{A}_{\alpha}=N(\operatorname{tr} \hat{M}) \mathbb{1} \tag{5.54}
\end{equation*}
$$

which is Eq. (5.39). Thus Eqs. (5.41) and (5.45) are valid for any definition of the discrete Wigner function in Gibbons et al. [53].

In this section, we have shown that the transition rates may be thought of as just as fundamental as the Hamiltonian. In fact, we do not need a Hamiltonian to describe the dynamics in discrete phase space. All we need is a set of transition rates.

## - Section 5.7

## The Transition Rate Constraint

In the last section, we saw how the Hamiltonian could be superseded by a set of transition rates. Legal transition rates and Hamiltonians uniquely determine each other. But it takes only $N^{2}-1$ real numbers to specify the Hamiltonian up to an additive constant, whereas the set of $r_{\alpha \gamma}$ 's would seem to contain $N^{2}\left(N^{2}-1\right) / 2$ real numbers: there is a value of $r$ for every pair of distinct points in phase space, and there are $N^{2}\left(N^{2}-1\right) / 2$ pairs of distinct points. Therefore, we must not be free to
specify each of these values of $r$ independently. There must be a constraint on the values.

To see a specific example of a set of $r$ values that is not allowed, consider the following proposed choice for the rates:

$$
\begin{equation*}
r_{\alpha \gamma}=\kappa\left(\delta_{\alpha, \gamma+\epsilon}-\delta_{\alpha, \gamma-\epsilon}\right), \tag{5.55}
\end{equation*}
$$

where $\kappa$ is a positive constant and $\epsilon$ is the fixed vector $\epsilon=(1,0)$. This set of rates would imply, for example, that a "position eigenstate," having non-zero Wigner function only along a vertical line in phase space, would, to first order in time, evolve by having some of its weight move one step to the right (at the expense of the column one step to the left, which picks up some negative weight). This is not a legal evolution. For one thing, the column at the left would pick up a negative marginal probability, which is not allowed. Also, the probability of the column at the right would grow linearly in time starting from zero, which is not possible. (The amplitude of a state can grow linearly in time.) So the transition rate defined in Eq. (5.55) is evidently one that will be ruled out by the constraint we are about to derive.

Noting that the last example is translationally invariant, we might wonder if it is generally illegal for a set of transition rates to have this property? Start by assuming that this is the case:

$$
\begin{equation*}
r_{\alpha, \gamma}=r_{\alpha+\delta, \gamma+\delta}=\frac{2}{\hbar} \sum_{\beta} \operatorname{Im}\left[\Gamma_{\alpha+\delta, \beta, \gamma+\delta}\right] H_{\beta}, \tag{5.56}
\end{equation*}
$$

where $\delta$ is some displacement. If $S$ is a symplectic transformation, therefore preserving areas, we know $\Gamma_{\alpha, \beta, \gamma}=\Gamma_{S(\alpha), S(\beta), S(\gamma)}$. Moreover, $\frac{2}{\hbar} \sum_{\beta} \operatorname{Im}\left[\Gamma_{\alpha+\delta, \beta, \gamma+\delta}\right] H_{\beta}=$
$\frac{2}{\hbar} \sum_{\beta} \operatorname{Im}\left[\Gamma_{\alpha, \beta-\delta, \gamma}\right] H_{\beta}$ which leads to

$$
\begin{equation*}
r_{\alpha, \gamma}=\frac{2}{\hbar} \sum_{\beta} \operatorname{Im}\left[\Gamma_{\alpha, \beta-\delta, \gamma}\right] H_{\beta} . \tag{5.57}
\end{equation*}
$$

We sum over $\delta$ and divide by $N^{2}$ on both sides of the equation above, leveraging the phase point operator resolution of identity in Eq. (4.68) to find

$$
\begin{align*}
r_{\alpha \gamma} & =\frac{2}{N^{2} \hbar} \sum_{\beta} \operatorname{Im}\left[\operatorname{Tr}\left(\hat{A}_{\alpha} \sum_{\delta} \hat{A}_{\beta-\delta} \hat{A}_{\gamma}\right)\right] H_{\beta}  \tag{5.58}\\
& =\frac{2}{N^{2} \hbar} \sum_{\beta} \operatorname{Im}\left[\operatorname{Tr}\left(\hat{A}_{\alpha} \hat{A}_{\gamma}\right)\right] H_{\beta}  \tag{5.59}\\
& =0 . \tag{5.60}
\end{align*}
$$

The only translationally invariant set of transition rates in the trivial set where each transition rate is equal to zero.

We now set about deriving the overall constraint. Given any proposed transition rate $r_{\alpha \gamma}$, one can use Eq. (5.45) to get a Hamiltonian function $H_{\beta}$. This Hamiltonian function can then be inserted into Eq. (5.33) to get a transition rate. This last transition rate must be the same as the one we started with. If it is not, then there is no Hamiltonian that can produce that transition rate.

Putting Eqs. (5.45) and (5.33) together in this way, we obtain the following constraint, which must hold for all values of $\alpha$ and $\gamma$ :

$$
\begin{equation*}
r_{\alpha \gamma}=\frac{2}{N^{2}} \sum_{\alpha^{\prime} \gamma^{\prime}}\left[\sum_{\beta} \operatorname{Im}\left(\Gamma_{\alpha \beta \gamma}\right) \operatorname{Im}\left(\Gamma_{\alpha^{\prime} \beta \gamma^{\prime}}\right)\right] r_{\alpha^{\prime} \gamma^{\prime}} \tag{5.61}
\end{equation*}
$$

For the case of odd prime $N$ with the Wigner function of Wootters (1987), we have

$$
\begin{equation*}
\operatorname{Im}\left(\Gamma_{\alpha \beta \gamma}\right)=\frac{1}{N} \sin \left[\frac{4 \pi}{N}\left(a_{1} b_{2}+b_{1} c_{2}+c_{1} a_{2}-a_{2} b_{1}-b_{2} c_{1}-c_{2} a_{1}\right)\right] . \tag{5.62}
\end{equation*}
$$

Inserting this expression into Eq. (5.61) and expanding the sine in exponentials, one can perform the sum over $\beta$ in the usual way. The constraint then comes out to be

$$
\begin{align*}
r_{\alpha \gamma}=\frac{1}{N^{2}} \sum_{\alpha^{\prime} \gamma^{\prime}} & \left\{\delta_{\alpha-\gamma, \alpha^{\prime}-\gamma^{\prime}} \cos \left[\frac{4 \pi}{N}\left(\langle\gamma, \alpha\rangle-\left\langle\gamma^{\prime}, \alpha^{\prime}\right\rangle\right)\right]\right.  \tag{5.63}\\
& \left.-\delta_{\alpha-\gamma, \gamma^{\prime}-\alpha^{\prime}} \cos \left[\frac{4 \pi}{N}\left(\langle\gamma, \alpha\rangle+\left\langle\gamma^{\prime}, \alpha^{\prime}\right\rangle\right)\right]\right\} r_{\alpha^{\prime} \gamma^{\prime}} \tag{5.64}
\end{align*}
$$

where $\langle\gamma, \alpha\rangle$ is the usual symplectic form. After completing the sum over $\gamma^{\prime}$ and letting $\delta=\gamma-\alpha$, we get

$$
\begin{equation*}
r_{\alpha, \alpha+\delta}=\frac{1}{N^{2}} \sum_{\alpha^{\prime}}\left(r_{\alpha^{\prime}, \alpha^{\prime}+\delta}-r_{\alpha^{\prime}, \alpha^{\prime}-\delta}\right) \cos \left[\frac{4 \pi}{N}\left\langle\delta, \alpha-\alpha^{\prime}\right\rangle\right] . \tag{5.65}
\end{equation*}
$$

This relation must hold for all values of $\alpha$ and $\delta$. Any set of transition rates that satisfies this constraint is allowed-its Hamiltonian function is given by Eq. (5.45)and any set of $r$ values that does not satisfy this constraint is not allowed.

Eq. (5.65) can be made simpler but first we make it uglier by replacing every $\delta$ with $\gamma-\alpha$ :

$$
\begin{equation*}
r_{\alpha \gamma}=\frac{1}{N^{2}} \sum_{\alpha^{\prime}}\left(r_{\alpha^{\prime}, \alpha^{\prime}+\gamma-\alpha}-r_{\alpha^{\prime}, \alpha^{\prime}+\alpha-\gamma}\right) \cos \left[\frac{4 \pi}{N}\left\langle\gamma-\alpha, \alpha-\alpha^{\prime}\right\rangle\right] . \tag{5.66}
\end{equation*}
$$

Inside the summation, there are two terms. For the first term, change the summation variable to $\zeta=\alpha^{\prime}-\alpha$. For the second term, change the summation variable to $\zeta^{\prime}=\alpha^{\prime}-\gamma$. This gives us

$$
\begin{equation*}
r_{\alpha \gamma}=\frac{1}{N^{2}}\left\{\sum_{\zeta} r_{\alpha+\zeta, \gamma+\zeta} \cos \left[\frac{4 \pi}{N}\langle\gamma-\alpha, \zeta\rangle\right]-\sum_{\zeta^{\prime}} r_{\gamma+\zeta^{\prime}, \alpha+\zeta^{\prime}} \cos \left[\frac{4 \pi}{N}\left\langle\gamma-\alpha, \zeta^{\prime}\right\rangle\right]\right\} . \tag{5.67}
\end{equation*}
$$

Renaming the summation variable in the second sum to $\zeta$, we get

$$
\begin{equation*}
r_{\alpha \gamma}=\frac{1}{N^{2}} \sum_{\zeta}\left(r_{\alpha+\zeta, \gamma+\zeta}-r_{\gamma+\zeta, \alpha+\zeta}\right) \cos \left[\frac{4 \pi}{N}\langle\gamma-\alpha, \zeta\rangle\right] \tag{5.68}
\end{equation*}
$$

Finally, if we allow ourselves to assume that $r_{\gamma+\zeta, \alpha+\zeta}=-r_{\alpha+\zeta, \gamma+\zeta}$, we can combine the two terms in the sum:

$$
\begin{equation*}
r_{\alpha \gamma}=\frac{2}{N^{2}} \sum_{\zeta} r_{\alpha+\zeta, \gamma+\zeta} \cos \left[\frac{4 \pi}{N}\langle\gamma-\alpha, \zeta\rangle\right] \tag{5.69}
\end{equation*}
$$

Note, that Eq. (5.69) cannot be used as the sole requirement on the $r$ 's. It must be in conjunction with the requirement that $r_{\gamma \alpha}=-r_{\alpha \gamma}$. In contrast, Eq (5.68) can be used as the sole requirement on the $r$ 's. Eq. (5.68) implies that $r_{\gamma \alpha}=-r_{\alpha \gamma}$.

## - Section 5.8

## Geometric Implications of the Constraint

### 5.8.1. Lines with Equal Transition Rate Magnitudes

We now prove that the co-linear transition rates generated by a specific displacement $(\gamma-\alpha)$ must be the same. (i.e. $r_{\alpha, \gamma}=r_{\alpha+n(\gamma-\alpha), \gamma+n(\gamma-\alpha)}$ for $\left.n \in \mathbb{Z} / p \mathbb{Z}\right)$. Begin with the constraint on the transitions rates and recall that $\langle\gamma-\alpha, \gamma-\alpha\rangle=0$ so that

$$
\begin{align*}
r_{\alpha, \gamma} & =\frac{1}{N^{2}} \sum_{\zeta}\left(r_{\alpha+\zeta, \gamma+\zeta}-r_{\gamma+\zeta, \alpha+\zeta}\right) \cos \left[\frac{4 \pi}{N}\langle\gamma-\alpha, \zeta\rangle\right]  \tag{5.70}\\
& =\frac{1}{N^{2}} \sum_{\zeta}\left(r_{\alpha+\zeta, \gamma+\zeta}-r_{\gamma+\zeta, \alpha+\zeta}\right) \cos \left[\frac{4 \pi}{N}\langle\gamma-\alpha, \zeta-n(\gamma-\alpha)\rangle\right] \tag{5.71}
\end{align*}
$$

where $n \in \mathbb{Z} / p \mathbb{Z}$. We use this requirement for $n$ in what follows. The summation over $\zeta$ is over the complete space so it can be re-expressed as a sum over some other
position offset by a displacement $n(\gamma-\alpha)$ such that $\zeta=\zeta^{\prime}+n(\gamma-\alpha)$ :

$$
\begin{align*}
r_{\alpha, \gamma}= & \frac{1}{N^{2}} \sum_{\zeta}\left(r_{\alpha+\zeta, \gamma+\zeta}-r_{\gamma+\zeta, \alpha+\zeta}\right) \cos \left[\frac{4 \pi}{N}\langle\gamma-\alpha, \zeta-n(\gamma-\alpha)\rangle\right]  \tag{5.72}\\
= & \frac{1}{N^{2}} \sum_{\zeta}\left(r_{\alpha+\zeta, \gamma+\zeta}-r_{\gamma+\zeta, \alpha+\zeta}\right) \cos \left[\frac{4 \pi}{N}\langle\gamma-\alpha, \zeta-n(\gamma-\alpha)\rangle\right]  \tag{5.73}\\
= & \frac{1}{N^{2}} \sum_{\zeta^{\prime}}\left(r_{\alpha+\zeta^{\prime}+n(\gamma-\alpha), \gamma+\zeta^{\prime}+n(\gamma-\alpha)}-r_{\gamma+\zeta^{\prime}+n(\gamma-\alpha), \alpha+\zeta^{\prime}+n(\gamma-\alpha)}\right)  \tag{5.74}\\
& \times \cos \left[\frac{4 \pi}{N}\left\langle\gamma-\alpha, \zeta^{\prime}\right\rangle\right]  \tag{5.75}\\
= & r_{\alpha+n(\gamma-\alpha), \gamma+n(\gamma-\alpha)} . \tag{5.76}
\end{align*}
$$

We again see the special role of lines in the phase space. We have shown that the transition rate between any two points on a line is the same as the transition rate between any two other points that have the same displacement relationship as the first pair.

What is left to investigate is how the cosine factor influences the transition rates. Varying $n$ in $(\alpha+n(\gamma-\alpha), \gamma+n(\gamma-\alpha))$ parameterizes a line in phase space, also determining its $N-1$ parallel lines. For each of the parallel lines, there is one unique area determined by the symplectic form. For points co-linear with the original pair, the cosine factor becomes 1 due to the symplectic form becoming zero, an expression of colinearity leading to an absence of area.

### 5.8.2. Points with Zero Flow

Before we set about understanding the sinusoidal nature of transition rate magnitudes, let us answer another question. Which point $\alpha$ will have no flow to it? A number of manipulations of the constraint equation leads to a pleasing result.

Let $\hat{H} \propto \hat{X}^{m} \hat{Z}^{n}+h . c .=\omega[-m n / 2][\hat{D}(m, n)+\hat{D}(-m,-n)]$ and $\eta=(m, n)$ so $\hat{D}(\eta)=\hat{D}(m, n)$. We know $\hat{H}$ is Hermitian so $H_{\beta}$ is a real function and it just scales
$\operatorname{Im}\left(\Gamma_{\alpha \beta \gamma}\right)$ in the equation for $r$ and can be pulled inside:

$$
\begin{align*}
r_{\alpha \gamma} & =\frac{2}{\hbar} \sum_{\beta} \operatorname{Im}\left(\Gamma_{\alpha \beta \gamma}\right) H_{\beta}  \tag{5.77}\\
& =\frac{2}{\hbar} \sum_{\beta} \operatorname{Im}\left(\frac{1}{N} \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{A}_{\beta} H_{\beta} \hat{A}_{\gamma}\right)\right)  \tag{5.78}\\
& =\frac{2}{N \hbar} \operatorname{Im}\left(\operatorname{Tr}\left(\hat{A}_{\alpha} \sum_{\beta} \hat{A}_{\beta} H_{\beta} \hat{A}_{\gamma}\right)\right)  \tag{5.79}\\
& =\frac{2}{N \hbar} \operatorname{Im}\left(\operatorname{Tr}\left(\hat{A}_{\alpha} \hat{H} \hat{A}_{\gamma}\right)\right) . \tag{5.80}
\end{align*}
$$

Focus on the trace for now and insert equations for $A$ 's and $\hat{H}$ :

$$
\begin{align*}
\operatorname{Tr}\left(\hat{A}_{\alpha} \hat{H} \hat{A}_{\gamma}\right)=\frac{1}{N^{2}} \operatorname{Tr} & {\left[\sum_{\beta} \omega[\langle\beta, \alpha\rangle] \hat{D}(\beta) \omega[-m n / 2]\right.}  \tag{5.81}\\
& \left.(\hat{D}(m, n)+\hat{D}(-m,-n)) \sum_{\delta} \omega[\langle\delta, \gamma\rangle] \hat{D}(\delta)\right] \tag{5.82}
\end{align*}
$$

We'll focus on the first term of $\hat{H}$ and ignore the Hermitian conjugate because as we will see, for sites $\alpha$ where the first term disappears, so will the second term. Using the multiplication rule for displacement operators we have

$$
\begin{equation*}
\hat{D}(\beta) \hat{D}(\eta) \hat{D}(\delta)=w\left[\frac{1}{2}\langle\eta, \beta\rangle+\frac{1}{2}\langle\delta,(\beta+\eta)\rangle\right] \hat{D}(\beta+\eta+\delta) . \tag{5.83}
\end{equation*}
$$

Plug this in to get

$$
\begin{align*}
& \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{X}^{m} \hat{Z}^{m} \hat{A}_{\gamma}\right)= \\
& \frac{1}{N^{2}} \operatorname{Tr}\left[\sum_{\beta, \delta} \omega\left[\langle\beta, \alpha\rangle+\langle\delta, \gamma\rangle-\frac{1}{2} m n+\frac{1}{2}\langle\eta, \beta\rangle+\frac{1}{2}\langle\delta, \beta+\eta\rangle\right] \hat{D}(\beta+\eta+\delta)\right] . \tag{5.84}
\end{align*}
$$

Pull the trace inside sum and recall $\operatorname{Tr}[\hat{D}(\beta+\eta+\delta)]=N \delta_{\beta+\eta+\delta, 0}$. This leaves us
with

$$
\begin{align*}
& \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{X}^{m} \hat{Z}^{m} \hat{A}_{\gamma}\right) \\
& \quad=\frac{1}{N} \sum_{\beta, \delta} \omega\left[\langle\beta, \alpha\rangle+\langle\delta, \gamma\rangle-\frac{1}{2} m n+\frac{1}{2}\langle\eta, \beta\rangle+\frac{1}{2}\langle\delta,(\beta+\eta)\rangle\right] \times \delta_{\beta+\eta+\delta, 0} . \tag{5.85}
\end{align*}
$$

Cancel the last term in the square brackets due to the Kronecker delta and regroup so that

$$
\begin{align*}
& \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{X}^{m} \hat{Z}^{n} \hat{A}_{\gamma}\right) \\
& \quad=\frac{1}{N} \omega\left[-\frac{1}{2} m n\right] \sum_{\beta, \delta} \omega\left[\left\langle\beta, \alpha-\frac{1}{2} \eta\right\rangle+\langle\delta, \gamma\rangle\right] \delta_{\beta+\eta+\delta, 0} \tag{5.86}
\end{align*}
$$

Use the Kronecker delta to insert $-(\beta+\eta)$ for $\delta$ :

$$
\begin{align*}
& \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{X}^{m} \hat{Z}^{n} \hat{A}_{\gamma}\right) \\
& \quad=\frac{1}{N} \omega\left[-\frac{1}{2} m n\right] \sum_{\beta, \delta} \omega\left[\left\langle\beta, \alpha-\frac{1}{2} \eta\right\rangle+\langle-\beta-\eta, \gamma\rangle\right] \delta_{\beta+\eta+\delta, 0}  \tag{5.87}\\
& \quad=\frac{1}{N} \omega\left[-\frac{1}{2} m n+\langle\gamma, \eta\rangle\right] \sum_{\beta, \delta} \omega\left[\left\langle\beta, \alpha-\gamma-\frac{1}{2} \eta\right\rangle\right] \delta_{\beta+\eta+\delta, 0}
\end{align*}
$$

Perform sum over $\delta$ noting that $\beta$ and $\eta$ are determined so $\sum_{\delta} \delta_{\beta+\eta+\delta, 0}=1$. Now we use one small Fourier relationship. Note that

$$
\begin{align*}
\sum_{\beta} \omega^{\beta \wedge \alpha} & =\sum_{b_{1}} \omega^{b_{1} a_{2}} \sum_{b_{2}} \omega^{-b_{2} a_{1}} \\
& =N^{2} \delta_{a_{1}, 0} \delta_{a_{2}, 0}  \tag{5.88}\\
& =N^{2} \delta_{\alpha, 0}
\end{align*}
$$

Leveraging this relationship, we simplify to

$$
\begin{align*}
\operatorname{Tr}\left(\hat{A}_{\alpha} \hat{X}^{m} \hat{Z}^{n} \hat{A}_{\gamma}\right) & =\frac{1}{N} \omega\left[-\frac{1}{2} m n+\langle\gamma, \eta\rangle\right] \sum_{\beta} \omega\left[\left\langle\beta, \alpha-\gamma-\frac{1}{2} \eta\right\rangle\right]  \tag{5.89}\\
& =N \omega\left[-\frac{1}{2} m n+\langle\gamma, \eta\rangle\right] \delta_{\alpha-\gamma-\frac{1}{2} \eta}  \tag{5.90}\\
& =N \omega\left[-\frac{1}{2} m n+\langle\alpha, \eta\rangle\right] \delta_{\alpha-\gamma-\frac{1}{2} \eta} . \tag{5.91}
\end{align*}
$$

We have arrived at an expression for $\operatorname{Tr}\left(\hat{A}_{\alpha} \hat{X}^{m} \hat{Z}^{n} \hat{A}_{\gamma}\right)$ where if we pick some point $\alpha$ with $\eta=(m, n)$ already specified then the possible nonzero $\gamma$ will be determined by the Kronecker delta. Our expression for the transition rate is now

$$
\begin{equation*}
r_{\alpha \gamma}=\frac{2}{\hbar} \operatorname{Im}(\omega[-m n / 2+\langle\alpha, \eta\rangle]) \delta_{\gamma-\alpha+\frac{1}{2} \eta, 0} \tag{5.92}
\end{equation*}
$$

There exists a choice of $\alpha$ such that $-m n / 2+a_{1} n-a_{2} m=0$ leaving us with $\operatorname{Im}(1)=$ $0=r_{\alpha \gamma}$. We can displace $\alpha$ by any multiple of $\eta$ and still retain this null result. All points $\alpha^{\prime}=\alpha+k \eta$ where $k \in \mathbb{Z}$ and $m n / 2=\langle\alpha, \eta\rangle$ will have zero transition rates.

### 5.8.3. Sinusoids in Transition Space

Equation (5.92) leads directly to an explanation of the sinusoidal oscillations in the magnitude of transition rates along certain directions of the transition space. Let $\bar{\alpha}$ be some phase space point that will never have any quasiprobability transitions (i.e. $\left.\langle\bar{\alpha}, \eta\rangle=\bar{a}_{1} n-\bar{a}_{2} m=m n / 2\right)$. If $\bar{\alpha}$ is displaced to $(\bar{\alpha}+\zeta)$, we will have a transition rate

$$
\begin{align*}
r_{\bar{\alpha}+\zeta, \gamma} & =\frac{2}{\hbar} \operatorname{Im}(\omega[-m n / 2+\langle\bar{\alpha}+\zeta, \eta\rangle]) \delta_{\gamma-\bar{\alpha}-\zeta+\frac{1}{2} \eta, 0}  \tag{5.93}\\
& =\frac{2}{\hbar} \operatorname{Im}(\omega[\langle\zeta, \eta\rangle]) \delta_{\gamma-\bar{\alpha}-\zeta+\frac{1}{2} \eta, 0} \tag{5.94}
\end{align*}
$$

We can now choose $\zeta$ in such a way so that the phase increases by any amount in $\mathbb{Z}(N)$. For example, the lowest frequency sinusoid will occur along a line pointing in the direction corresponding to a displacement $\zeta$ where $\langle\zeta, \eta\rangle=1$. On the discrete line, we will see a single period of the sinusoid. Two periods will occur for $\zeta$ such that $\langle\zeta, \eta\rangle=2$. More generally, pick any line in the phase space and calculate the displacement $\hat{D}_{\zeta}$ that generates the line. The magnitudes of the rates with which quasiprobability transitions into points on this line can be described by the discrete sinusoid covering $\langle\zeta, \eta\rangle$ periods across the $N$ points ordered by successive application of the displacement $\hat{D}_{\zeta}$. The symplectic form and resulting area is the determining structure for dynamics on the discrete phase space.

### 5.8.4. Hamiltonian to Transition Rates: A Three-step Recipe

It is remarkable how a Hamiltonian function can encode so much information in both classical Hamiltonian mechanics and quantum mechanics. It is now also possible to see how this remarkable compactification of information also occurs for the transition rates on phase space. We can now give three rules that allow us to calculate any possible transition rate in phase space for a monomial $\hat{X}^{m} \hat{Z}^{n}$ where the point $\eta=$ $(m, n)$ :
(1) Identify points $\bar{\alpha}=\left(\bar{a}_{1}, \bar{a}_{2}\right)$ to which no quasiprobability can transition via the equation $\bar{a}_{1} n-\bar{a}_{2} m=m n / 2$,
(2) For any point $\bar{\alpha}+\zeta$, possible transition rates of quasiprobability into this point will have magnitude $\frac{2}{\hbar} \sin \left(\frac{2 \pi}{N}\langle\zeta, \eta\rangle\right)$, and
(3) points $\gamma$ from which quasiprobability can be transferred with magnitude specified in rule (2) must satisfy the equality $\gamma=\bar{\alpha}+\zeta \pm \frac{1}{2} \eta$. The sign chosen indicates a factor of $\pm 1$ multiplied by the rate from rule (2).

Adherence to these rules allows one to convert any tuple ( $m, n$ ) corresponding to a displacement $\hat{D}(m, n)=\hat{D}_{\eta}$ and write out the complete set of transition rates that determines the phase space dynamics for that displacement.

Complicated dynamics will be described by Hamiltonians involving multiple terms stemming from different monomials and their Hermitian conjugates. However, If a Hamiltonian includes just one, we can make a symplectic change of the basis we assign to the horizontal axis in phase space so that such a system will look just like the free particle on a ring we studied in section 5.3. A monomial $\hat{X}^{k} \hat{Z}^{l}$ is proportional to $S(k, l \mid m, n) \hat{X}[S(k, l \mid m, n)]^{\dagger}=\omega\left(2^{-1} k l\right) \hat{X}^{k} \hat{Z}^{l}=\hat{D}(k, l)$ as shown in Eq. (4.52). The state for which this operator plays the role shift operator, is $|\hat{x}(m, n) ; j\rangle=$ $S(k, l \mid m, n)|\hat{x} ; j\rangle$ found in Eq. (4.55). Let us call this set of states our horizontal 'position' basis and we can return to the picture of quasiprobabilities transferring back and forth 'across the ring.'

## Section 5.9

## Projection onto Valid Transition Rates

Recall the general constraint on sets of transitions rates $r_{\alpha \gamma}$, given by Eq. (5.61):

$$
\begin{equation*}
r_{\alpha \gamma}=\frac{2}{N^{2}} \sum_{\alpha^{\prime} \gamma^{\prime}}\left[\sum_{\beta} \operatorname{Im}\left(\Gamma_{\alpha \beta \gamma}\right) \operatorname{Im}\left(\Gamma_{\alpha^{\prime} \beta \gamma^{\prime}}\right)\right] r_{\alpha^{\prime} \gamma^{\prime}} \tag{5.95}
\end{equation*}
$$

which is valid for every quantum net in the formulation of Gibbons et al. [53]. We can write this equation in matrix form:

$$
\begin{equation*}
\vec{r}=R \vec{r}, \tag{5.96}
\end{equation*}
$$

or in terms of components,

$$
\begin{equation*}
r_{\alpha \gamma}=\sum_{\alpha^{\prime} \gamma^{\prime}} R_{\alpha \gamma, \alpha^{\prime} \gamma^{\prime}} r_{\alpha^{\prime} \gamma^{\prime}} \tag{5.97}
\end{equation*}
$$

where the matrix $R$ is defined by

$$
\begin{equation*}
R_{\alpha \gamma, \alpha^{\prime} \gamma^{\prime}}=\frac{2}{N^{2}} \sum_{\beta} \operatorname{Im}\left(\Gamma_{\alpha \beta \gamma}\right) \operatorname{Im}\left(\Gamma_{\alpha^{\prime} \beta \gamma^{\prime}}\right) . \tag{5.98}
\end{equation*}
$$

A pair $\alpha \gamma$ is thought of as a single matrix index - we can picture $R$ as an $N^{4} \times N^{4}$ matrix and $\vec{r}$ as vector with $N^{4}$ entries. In this section, we want to show that $R$ is a projection operator onto a space of dimension $N^{2}-1$, namely, the space of legal $\vec{r}$-vectors. Thus, if we apply $R$ to any vector of potential transition rates (not necessarily legal), the result will be a vector of legal transition rates.

The main thing we need to show here is that $R$ is a projection operator, that is, $R^{2}=R$. By inspection of Eq. (5.98), we already know that $R$ is a symmetric real matrix - it will be a projection operator if it is idempotent. The fact that Eq. (5.96) is valid if and only if $\vec{r}$ is legal will then tell us that the space that $R$ projects onto is the $\left(N^{2}-1\right)$-dimensional space of legal $\vec{r}$-vectors. We begin by proving the following fact:

$$
\begin{equation*}
\frac{2}{N^{2}} \sum_{\alpha \gamma} \operatorname{Im}\left(\Gamma_{\alpha \beta \gamma}\right) \operatorname{Im}\left(\Gamma_{\alpha \beta^{\prime} \gamma}\right)=-\frac{1}{N^{2}}+\delta_{\beta \beta^{\prime}} \tag{5.99}
\end{equation*}
$$

Write $\Gamma$ in terms of the $A$ operators and expand the imaginary parts. The left-hand side of Eq. (5.99) can be written as

$$
\begin{equation*}
-\frac{1}{2 N^{4}} \sum_{\alpha \gamma}\left[\operatorname{Tr}\left(\hat{A}_{\alpha} \hat{A}_{\beta} \hat{A}_{\gamma}\right)-\operatorname{Tr}\left(\left(\hat{A}_{\alpha} \hat{A}_{\gamma} \hat{A}_{\beta}\right)\right]\left[\operatorname{Tr}\left(\hat{A}_{\alpha} \hat{A}_{\beta^{\prime}} \hat{A}_{\gamma}\right)-\operatorname{Tr}\left(\left(\hat{A}_{\alpha} \hat{A}_{\gamma} \hat{A}_{\beta^{\prime}}\right)\right]\right.\right. \tag{5.100}
\end{equation*}
$$

The multiplication gives four terms inside the sum. The sum over $\alpha$ and $\gamma$ is performed
for two representative terms:

$$
\begin{align*}
\sum_{\alpha \gamma} \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{A}_{\beta} \hat{A}_{\gamma}\right) \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{A}_{\beta^{\prime}} \hat{A}_{\gamma}\right) & =\sum_{\gamma} \operatorname{Tr}\left\{\left[\sum_{\alpha} \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{A}_{\beta} \hat{A}_{\gamma}\right) \hat{A}_{\alpha}\right] \hat{A}_{\beta^{\prime}} \hat{A}_{\gamma}\right\} \\
& =N \sum_{\gamma} \operatorname{Tr}\left(\hat{A}_{\beta} \hat{A}_{\gamma} \hat{A}_{\beta^{\prime}} \hat{A}_{\gamma}\right)  \tag{5.101}\\
& =N \operatorname{Tr}\left[\hat{A}_{\beta} \sum_{\gamma} \hat{A}_{\gamma} \hat{A}_{\beta^{\prime}} \hat{A}_{\gamma}\right] \\
& =N^{2}\left(\operatorname{Tr} \hat{A}_{\beta}\right)\left(\operatorname{Tr} \hat{A}_{\beta^{\prime}}\right)=N^{2}
\end{align*}
$$

Eq. (5.54) was used to reach the last line. The other representative term is

$$
\begin{align*}
\sum_{\alpha \gamma} \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{A}_{\gamma} \hat{A}_{\beta}\right) \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{A}_{\beta^{\prime}} \hat{A}_{\gamma}\right) & =\sum_{\gamma} \operatorname{Tr}\left\{\left[\sum_{\alpha} \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{A}_{\gamma} \hat{A}_{\beta}\right) \hat{A}_{\alpha}\right] \hat{A}_{\beta^{\prime}} \hat{A}_{\gamma}\right\} \\
& =N \sum_{\gamma} \operatorname{Tr}\left(\hat{A}_{\gamma} \hat{A}_{\beta} \hat{A}_{\beta^{\prime}} \hat{A}_{\gamma}\right)  \tag{5.102}\\
& =N \operatorname{Tr}\left[\sum_{\gamma} \hat{A}_{\gamma} \hat{A}_{\beta} \hat{A}_{\beta^{\prime}} \hat{A}_{\gamma}\right] \\
& =N^{2} \operatorname{Tr}\left[\operatorname{Tr}\left(\hat{A}_{\beta} \hat{A}_{\beta^{\prime}}\right) \mathbb{1}\right]=N^{4} \delta_{\beta \beta^{\prime}}
\end{align*}
$$

The other two terms are similar to the two just summed. Putting everything together, we get Eq. (5.99).

Now, we want to show that

$$
\begin{equation*}
\sum_{\alpha^{\prime} \gamma^{\prime}} R_{\alpha \gamma, \alpha^{\prime} \gamma^{\prime}} R_{\alpha^{\prime} \gamma^{\prime}, \alpha^{\prime \prime} \gamma^{\prime \prime}}=R_{\alpha \gamma, \alpha^{\prime \prime} \gamma^{\prime \prime}} \tag{5.103}
\end{equation*}
$$

We begin by writing the left-hand side in terms of $\operatorname{Im}\left(\Gamma_{\alpha \beta \gamma}\right)$ :

$$
\begin{equation*}
\frac{4}{N^{4}} \sum_{\alpha^{\prime} \gamma^{\prime}}\left(\sum_{\beta} \operatorname{Im}\left(\Gamma_{\alpha \beta \gamma}\right) \operatorname{Im}\left(\Gamma_{\alpha^{\prime} \beta \gamma^{\prime}}\right) \sum_{\beta^{\prime}} \operatorname{Im}\left(\Gamma_{\alpha^{\prime} \beta^{\prime} \gamma^{\prime}}\right) \operatorname{Im}\left(\Gamma_{\alpha^{\prime \prime} \beta^{\prime} \gamma^{\prime \prime}}\right)\right) . \tag{5.104}
\end{equation*}
$$

Now we use the lemma to do the sum over $\alpha^{\prime}$ and $\gamma^{\prime}$. This gives us

$$
\begin{equation*}
\frac{2}{N^{2}} \sum_{\beta \beta^{\prime}} \operatorname{Im}\left(\Gamma_{\alpha \beta \gamma}\right)\left(-\frac{1}{N^{2}}+\delta_{\beta \beta^{\prime}}\right) \operatorname{Im}\left(\Gamma_{\alpha^{\prime \prime} \beta^{\prime} \gamma^{\prime \prime}}\right) \tag{5.105}
\end{equation*}
$$

The term with $1 / N^{2}$ yields zero (the imaginary part of $\Gamma$ vanishes when we sum over one of the indices). So we are left with

$$
\begin{equation*}
\frac{2}{N^{2}} \sum_{\beta} \operatorname{Im}\left(\Gamma_{\alpha \beta \gamma}\right) \operatorname{Im}\left(\Gamma_{\alpha^{\prime \prime} \beta \gamma^{\prime \prime}}\right), \tag{5.106}
\end{equation*}
$$

which equals $R_{\alpha \gamma, \alpha^{\prime \prime} \gamma^{\prime \prime}}$. This is what we wanted to show.

To summarize, if we want to test a prospective transition-rate vector $\vec{r}$ to see whether it is legal, we check to see if it is an eigenvector of $R$ with eigenvalue 1. Moreover, if we apply the matrix $R$ to any vector, the result will always be a legal transition-rate vector. If the result turns out to be the zero vector, then it is not a very interesting legal vector, but it is legal.

- Section 5.10


## Summary

In this chapter we introduced a discrete Wigner function and examined the more useful corresponding transition rates. We found interesting patterns in the space of transition rates which have the unexpected property of being able to take negative values. Owing to the clever dance in phase space which includes negative transition rates and negative quasiprobabilities of the Wigner function, the marginal probabilities remain positive. A Hamiltonian description was shown to be on equal footing with a description of dynamics encoded in a set of transition rates. We made an exhaustive investigation of the geometric constraints in transition space that are im-
plied by the analytical constraint in Eq. (5.68). Up to a scalar constant, we now have a good understanding of every possible dynamical evolution that occur in the phase space. Finally, we demonstrated a procedure to check whether a set of transition rates corresponds to legal quantum dynamics.

## Chapter 6

## General Quantum Processes in Discrete Phase Space

Our discussion of the dynamics of closed quantum systems gave us a clear geometric understanding of how finite quantum system evolve in phase space. However, such idealized scenarios are often a far-flung dream from the reality in the lab. Interactions between the object of study and its surroundings will inevitably make their presence felt at some point as correlations build up between what we call the system and its bath.

For this reason, it is important to delve into the description of the noisy dynamics of such open quantum systems in discrete phase space. In this chapter, we build off the theory of Kraus operators to find general transition probabilities in the phase space. On the way, we briefly review the description of classical probabilistic processes. A new structure function arises in this setting and it adheres to some interesting geometrical constraints. Again, we codify when a set of transition rates constitutes legal quantum dynamics. Making the Markovian approximation, we study Lindbladian dynamics.

## - Section 6.1

## Open Quantum Systems

In this section we review how one may treat the dynamical processes of a quantum system interacting with a noisy environment or an open quantum system. They are more precisely described as a principal system which becomes correlated with the behavior of external systems which we call the environment. We draw from Nielsen and Chuang [100] and John Preskill's online lecture notes [109].

The physically motivated approach relies on the familiar unitary dynamics of closed quantum systems. In the last chapter, we studied the map of the von Neumann equation into phase space. For a time-independent Hamiltonian $\hat{H}$, a solution to that equation is the unitary evolution $\hat{\rho}(t)=\exp (-i \hat{H} t / \hbar) \hat{\rho}(0) \exp (i \hat{H} t / \hbar)$. We now imagine an initial system and environment product state $\hat{\rho} \otimes \hat{\rho}_{\text {env }}$ which, in totality, undergoes some unitary evolution. When we only look at the state of the system after some time without gaining any information about the environment, we average over all possible environmental states. Expressed mathematically, this is equivalent to a partial trace over the environment over the evolved product state:

$$
\begin{equation*}
\mathcal{E}(\hat{\rho})=\operatorname{Tr}_{\text {env }}\left[\hat{U}\left(\hat{\rho} \otimes \hat{\rho}_{\text {env }}\right) \hat{U}^{\dagger}\right] . \tag{6.1}
\end{equation*}
$$

The constraint of initializing the system in a product state might seem nonphysical in some instances but in many cases, we can imagine an experimentalist who is able to prepare an initial state of the system which is uncorrelated with the environment. Nevertheless, we will show how to relax this condition in a future section.

The Kraus representation is a natural extension of the last treatment where we construct dynamical map using operators only on the principal systems Hilbert space. It is simply a basis-dependent explicit restatement of the Equation (6.1). Let $\left|b_{j}\right\rangle$ be
an orthonormal basis for the environmental state space and let $\left|b_{0}\right\rangle\left\langle b_{0}\right|$ be the initial state of the environment. This initial state assumption leads to general results because we can always introduce an auxiliary fictitious environment to purify the state of the original environment without effecting the dynamics. We rewrite Equation (6.1) as

$$
\begin{align*}
\mathcal{E}(\rho) & =\sum_{j}\left\langle b_{j}\right| \hat{U}[(\hat{\rho} \otimes \hat{\rho} \mathrm{env})] \hat{U}^{\dagger}\left|b_{j}\right\rangle  \tag{6.2}\\
& =\sum_{j} \hat{B}_{j} \hat{\rho} \hat{B}_{j}^{\dagger} \tag{6.3}
\end{align*}
$$

where $\hat{B}_{j} \equiv\left\langle b_{j}\right| \hat{U}\left|b_{0}\right\rangle$ operates solely on the system state space. Equation (6.3) is known as the Kraus or operator-sum representation and will be of great use to us due to its calculational practicality. The operators $\left\{\hat{B}_{j}\right\}$ are known as the Kraus operators for the quantum operation $\mathcal{E}$.

For any quantum operation that maps one density matrix to another, there is an important constraint that the Kraus operators must satisfy. This constraint is called the completeness relation and stems from the fact that all density matrices have a trace equal to 1 ,

$$
\begin{equation*}
1=\operatorname{Tr}[\mathcal{E}(\hat{\rho})]=\operatorname{Tr}\left[\sum_{j} \hat{B}_{j} \hat{\rho} \hat{B}_{j}^{\dagger}\right]=\operatorname{Tr}\left[\sum_{j} \hat{B}_{j}^{\dagger} \hat{B}_{j} \hat{\rho}\right] \tag{6.4}
\end{equation*}
$$

This will remain true no matter what state $\hat{\rho}$ is so it must be that

$$
\begin{equation*}
\sum_{j} \hat{B}_{j}^{\dagger} \hat{B}_{j}=\mathbb{1} . \tag{6.5}
\end{equation*}
$$

Quantum operations that satisfy this equation are trace-preserving by construction. We can relax this by requiring that $\sum_{j} \hat{B}_{j}^{\dagger} \hat{B}_{j} \leq \mathbb{1}$ which gives us non-trace-preserving quantum operations; in these processes a measurement obtains extra information
about what occurred in the process. We shall restrict ourselves to the trace-preserving arena.

Briefly, we can show that the Kraus representation is only unique up to a unitary transformation. This stems from the fact that we may perform our partial trace over any basis of the environment that we like. Suppose we rotate our initial set of basis states so that we now have

$$
\begin{equation*}
\left|c_{k}\right\rangle=\sum_{j}\left|b_{j}\right\rangle \hat{V}_{k j}, \tag{6.6}
\end{equation*}
$$

where $V$ is some unitary operator. The new Kraus operators will obey the relationship

$$
\begin{equation*}
\hat{C}_{k}=\sum_{j} \hat{V}_{k j} \hat{B}_{j} \tag{6.7}
\end{equation*}
$$

## - Section 6.2

## Discrete Phase Space Dynamics

The Kraus operator representation can allow us to create more general dynamical maps in discrete phase space. The path is straightforward and we have walked it before in section 5.1. Express the operators in Eq. (6.3) as expansions over the phase point operators and denote $\hat{\rho}^{\prime}=\mathcal{E}(\hat{\rho})=\sum_{\alpha} W_{\alpha}^{\prime} \hat{A}_{\alpha} . \hat{\rho}^{\prime}=\sum_{j} \hat{B}_{j} \hat{\rho} \hat{B}_{j}^{\dagger}$ then becomes

$$
\begin{equation*}
\sum_{\alpha} W_{\alpha}^{\prime} \hat{A}_{\alpha}=\sum_{j}\left(\sum_{\beta} B_{j \beta} \hat{A}_{\beta}\right)\left(\sum_{\gamma} W_{\gamma} \hat{A}_{\gamma}\right)\left(\sum_{\delta} \bar{B}_{j \delta} \hat{A}_{\delta}\right) \tag{6.8}
\end{equation*}
$$

where $\bar{B}_{j \delta}$ denotes the complex conjugate of $B_{j \delta}$. It is understood that $\delta$ appearing in a subscript denotes a point in phase space and is not a Kronecker delta. Using the orthonormality of the phase point operators under the trace operation, we can
rearrange this equation so that we have

$$
\begin{equation*}
W_{\alpha}^{\prime}=\frac{1}{N} \sum_{\beta \gamma \delta}\left(\sum_{j} B_{j \beta} \bar{B}_{j \delta}\right) \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{A}_{\beta} \hat{A}_{\gamma} \hat{A}_{\delta}\right) W_{\gamma} \tag{6.9}
\end{equation*}
$$

Equations involving products of operators lead to phase space equations involving the three-point structure function $\Gamma_{\alpha \beta \gamma}$. We now have the situation with a product of four operators which leads us to define the four-point structure function as $\Xi_{\alpha \beta \gamma \delta}=$ $\frac{1}{N} \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{A}_{\beta} \hat{A}_{\gamma} \hat{A}_{\delta}\right)$. The sum over $j$ is in some sense a dynamical coupling with many paths $j$ between the points $\beta$ and $\delta$ which we call the Kraus coupling $\mathcal{B}_{\beta \delta}=\sum_{j} B_{j \beta} \bar{B}_{j \delta}$. Remember that $B_{j \beta}=\frac{1}{N} \operatorname{Tr}\left(\hat{B}_{j} \hat{A}_{\beta}\right)$. With these definitions, we can rewrite equation (6.9) as

$$
\begin{equation*}
W_{\alpha}^{\prime}=\sum_{\gamma}\left(\sum_{\beta \delta} \mathcal{B}_{\beta \delta} \Xi_{\alpha \beta \gamma \delta}\right) W_{\gamma} . \tag{6.10}
\end{equation*}
$$

If we now define

$$
\begin{equation*}
P_{\alpha \gamma} \equiv \sum_{\beta \delta} \mathcal{B}_{\beta \delta} \Xi_{\alpha \beta \gamma \delta} \tag{6.11}
\end{equation*}
$$

we can display the dynamical map in phase space as

$$
\begin{equation*}
W_{\alpha}^{\prime}=\sum_{\gamma} P_{\alpha \gamma} W_{\gamma} \tag{6.12}
\end{equation*}
$$

$P_{\alpha \gamma}$ contains all the information about how a quasiprobability distribution $W_{\gamma}$ may be mapped.

One difference between the description of quantum dynamics in phase space and that in Hilbert space is that unitary transformations will change the Kraus operator description (or unraveling) in Hilbert space while the $P$ 's do not change. This is due to the fact that all operators in the $P$ 's occur inside a trace operator so the unitaries will cancel amongst themselves. However, as we explored with the group symmetries in Chapter 4, the phase space representation is only unique up to automorphisms of
the displacement operators.
It is interesting to draw the comparison between the continuous dynamics we studied in the last chapter. In particular, for Hamiltonian systems we know that

$$
\begin{equation*}
\frac{\partial W_{\alpha}}{\partial t}=\sum_{\gamma} r_{\alpha \gamma} W_{\gamma} \tag{6.13}
\end{equation*}
$$

What is the relationship between this equation and the last? How are $P_{\alpha \gamma}$ and $r_{\alpha \gamma}$ related? Before answering these questions, let us make a brief detour to solidify our foundations.

## Section 6.3

## Comparison with Classical Stochastic Processes

One of our main tasks is to distinguish between quantum and classical dynamics on discrete phase space. An understanding of classical probabilistic processes is a prerequisite for providing this comparison. A master equation describes the phenomenological gain and loss of probabilities for the discrete states of some Markov process [51]. It is a common form of equation that will allow us to make some comparisons between general probabilistic maps and continuous stochastic processes. As a suggestion of the comparison to come, let us denote the states of some classical Markov process with lower case greek letters. $G_{\alpha}$ will denote the probability of the system being found in the state $\alpha$. After some process happens, $G_{\alpha}$ will be mapped to some other state $G_{\alpha}^{\prime}$ by

$$
\begin{equation*}
G_{\alpha}^{\prime}=\sum_{\gamma} T_{\alpha \gamma} G_{\gamma}, \tag{6.14}
\end{equation*}
$$

where $T_{\alpha \gamma}$ is the conditional probability that state $\gamma$ will be mapped to state $\alpha$ in one iteration of the mapping process. Since a system in state $\gamma$ must go to some final
state, we must have

$$
\begin{equation*}
\sum_{\alpha} T_{\alpha \gamma}=1 \tag{6.15}
\end{equation*}
$$

There are a few ways to interpret Eq. (6.14). One particularly illustrative form encapsulates the new probability of being in state $\alpha$ as the sum of the old probability of being in state $\alpha$ plus the probability of transitioning into state $\alpha$ from $\gamma$ minus the probability of transitioning from state $\alpha$ to some other state $\gamma$ :

$$
\begin{equation*}
G_{\alpha}^{\prime}=G_{\alpha}+\sum_{\gamma \neq \alpha} T_{\alpha \gamma} G_{\gamma}-\sum_{\gamma \neq \alpha} T_{\gamma \alpha} G_{\alpha} \tag{6.16}
\end{equation*}
$$

Imagine now that a general probabilistic map was performed at some distinct time step $\Delta$ with $T_{\alpha \gamma}(\Delta)$ the conditional probability that state $\gamma$ will be mapped to state $\alpha$ in time $\Delta$. We now rewrite Eq. (6.14) as

$$
\begin{equation*}
G_{\alpha}(t+\Delta)=\sum_{\gamma} T_{\alpha \gamma}(\Delta) G_{\gamma}(t) \tag{6.17}
\end{equation*}
$$

If the evolution that this equation describes is continuous, we may take an infinitesimal limit $\Delta \rightarrow d t$ where the dependence of $T$ on $\Delta$ is a Taylor expansion of first order, i.e. linear:

$$
T_{\alpha \gamma}(d t)= \begin{cases}S_{\alpha \gamma} d t, & \alpha \neq \gamma  \tag{6.18}\\ 1-\sum_{\beta \neq \gamma} S_{\beta \gamma} d t, & \alpha=\gamma\end{cases}
$$

where $S_{\alpha \gamma}$ is the transition rate from state $\gamma$ to state $\alpha$. If the system starts in the state $\gamma$, the probability of staying there (over a time $d t$ ) is $1-\sum_{\beta \neq \gamma} S_{\beta \gamma} d t$, and the probability of moving to the state $\alpha$ is $S_{\alpha \gamma} d t$. Encoded is the fact that there can be no transition without passage of time. In the limit of $d t \rightarrow 0$, we obtain from Eq.
(6.17) the master equation [134]

$$
\begin{equation*}
\frac{\mathrm{d} G_{\alpha}(t)}{\mathrm{d} t}=\sum_{\gamma}\left(S_{\alpha \gamma} G_{\gamma}(t)-S_{\gamma \alpha} G_{\alpha}(t)\right) \tag{6.19}
\end{equation*}
$$

The transition rate of a state into itself is defined as $S_{\gamma \gamma}=-\sum_{\beta \neq \gamma} S_{\beta \gamma}$. For an ordinary probabilistic process, $S_{\gamma \gamma}$ is negative and each $S_{\alpha \gamma}$ is non-negative. Therefore Eq. (6.18) can be rewritten as

$$
\begin{equation*}
T_{\alpha \gamma}=\delta_{\alpha \gamma}+S_{\alpha \gamma} d t \tag{6.20}
\end{equation*}
$$

Furthermore, Eq. (6.19) can be rewritten as

$$
\begin{equation*}
\frac{d G_{\alpha}}{d t}=\sum_{\gamma} S_{\alpha \gamma} G_{\gamma} \tag{6.21}
\end{equation*}
$$

Notice the close resemblance between the discrete dynamical maps described by equations (6.12) and (6.14) and also the continuous dynamical equations (6.13) and (6.21). Pairwise, these equations take the same form. We can also write the quantum version of Eq. (6.20) as

$$
\begin{equation*}
P_{\alpha \gamma}=\delta_{\alpha \gamma}+r_{\alpha \gamma} d t \tag{6.22}
\end{equation*}
$$

Going from the classical setting where we have probabilities, transition rates, and transition probabilities, we find similar players in the quantum realm which are named with the prefix 'quasi-' to their classical partners. Of course we have just been using the phrase 'transition rate' in the last chapter because it was clear what domain we were describing. How do these three quantities differ from the classical to the quantum setting? It is well known that the Wigner function can take negative values and this is the most obvious difference between the classical case. What we have shown is the the transition rates, and the more general transition probabilities, may
also take negative values when describing quantum dynamics on phase space.
Digging a little deeper into the relationship between the $r$ 's and $P$ 's, we know that for reversible dynamics described with unitary operators in Hilbert space, the $r$ 's can be understood as antisymmetric $N^{2} \times N^{2}$ matrices indexed by phase space points. Using the definition for the $P$ 's in Eq. (6.11), one can derive the transition probabilities for some unitary transformation. We will denote the transition probabilities corresponding to such a reversible process as $Z_{\alpha \gamma}$. One finds

$$
\begin{align*}
Z_{\alpha \gamma} & =\frac{1}{N} \operatorname{Tr}\left(\hat{U}^{\dagger} \hat{A}_{\alpha} \hat{U} \hat{A}_{\gamma}\right)  \tag{6.23}\\
& =\sum_{\beta \delta} U_{\beta} \bar{U}_{\delta} \Xi_{\alpha \beta \gamma \delta}, \tag{6.24}
\end{align*}
$$

a conclusion Wootters also reached in his original work [143]. Following Wootters, notice that

$$
\begin{align*}
Z_{\alpha \beta} \hat{A}_{\beta} & =\frac{1}{N} \operatorname{Tr}\left(\hat{U}^{\dagger} \hat{A}_{\alpha} \hat{U} \hat{A}_{\gamma}\right) \hat{A}_{\beta}  \tag{6.25}\\
& =\hat{U}^{\dagger} \hat{A}_{\alpha} \hat{U}
\end{align*}
$$

where we used the fact that the first line includes an expansion of the resulting operator. Such unitary transformations leave unchanged the trace of a product of the A's which occur in $\Gamma_{\alpha \beta \gamma}$. We can use this fact to show that Z is an orthogonal
matrix:

$$
\begin{align*}
Z_{\alpha \rho} Z_{\beta \rho} & =Z_{\alpha \rho} Z_{\beta \sigma} \delta_{\rho \sigma} \\
& =\frac{1}{N} Z_{\alpha \rho} Z_{\beta \sigma} \sum_{\tau} \Gamma_{\rho \sigma \tau} \\
& =\frac{1}{N} \sum_{\gamma} Z_{\alpha \rho} Z_{\beta \sigma} Z_{\gamma \tau} \Gamma_{\rho \sigma \tau}  \tag{6.26}\\
& =\frac{1}{N} \sum_{\gamma} \Gamma_{\alpha \beta \gamma} \\
& =\delta_{\alpha \beta} .
\end{align*}
$$

We can now understand why the $r$ 's had to be antisymmetric in a different light. The P's form a Lie group with time as the continuous parameter. This is a Lie group of $N^{2} \times N^{2}$ orthogonal matrices. Each Lie group is conveniently studied by its group of generators living in the tangent space to the identity element on the manifold, the Lie algebra. For orthogonal matrices, the Lie algebra is spanned by a group of antisymmetric (or skew-symmetric) matrices. The exponential map generates our Lie group of $P$ 's from its Lie algebra consisting of the $r$ 's.

- Section 6.4


## The Four-point Structure Function $\Xi_{\alpha \beta \gamma \delta}$

Now we dig into the form of the four-point structure function $\Xi_{\alpha \beta \gamma \delta}$. The three-point structure function had a pleasing relation to areas of triangles in phase space. Will we find something similar here?

We can start by expressing the phase point operators in terms of displacement
operators using the Fourier relationship (4.69) $\Xi_{\alpha \beta \gamma \delta}$ :

$$
\begin{align*}
\Xi_{\alpha \beta \gamma \delta}= & \frac{1}{N} \operatorname{tr}\left(\hat{A}_{\alpha} \hat{A}_{\beta} \hat{A}_{\gamma} \hat{A}_{\delta}\right) \\
= & \frac{1}{N^{5}} \operatorname{Tr}\left[\sum_{\kappa \lambda \mu \nu} \hat{D}(\kappa) \omega[\langle\kappa, \alpha\rangle] \hat{D}(\lambda) \omega[\langle\lambda, \beta\rangle]\right.  \tag{6.27}\\
& \quad \times \hat{D}(\mu) \omega(\langle\mu, \gamma\rangle) \hat{D}(\nu) \omega(\langle\nu, \delta\rangle)]
\end{align*}
$$

By iterating the displacement operator multiplication rule (4.29) we arrive at

$$
\begin{align*}
\hat{D}(\kappa) \hat{D}(\lambda) \hat{D}(\mu) \hat{D}(\nu)=\hat{D}(\kappa+\lambda+\mu+\nu) \omega[ & \frac{1}{2}(\langle\nu, \kappa+\lambda+\mu\rangle  \tag{6.28}\\
& +\langle\mu, \kappa+\lambda\rangle+\langle\lambda, \kappa\rangle)]
\end{align*}
$$

We can pull the trace into the summation in $\Xi_{\alpha \beta \gamma \delta}$ so we have

$$
\begin{align*}
\operatorname{Tr} \hat{D}(\kappa+\lambda+\mu+\nu) & =\omega\left[\frac{1}{2}\left(k_{1}+l_{1}+m_{1}+n_{1}\right)\left(k_{2}+l_{2}+m_{2}+n_{2}\right)\right] \\
& \times \operatorname{Tr}\left[\hat{X}^{k_{1}+l_{1}+m_{1}+n_{1}} \hat{Z}^{k_{2}+l_{2}+m_{2}+n_{2}}\right] \\
& =\omega\left[\frac{1}{2}\left(k_{1}+l_{1}+m_{1}+n_{1}\right)\left(k_{2}+l_{2}+m_{2}+n_{2}\right)\right]  \tag{6.29}\\
& \times N \delta_{k_{1}+l_{1}+m_{1}+n_{1}, N} \delta_{k_{2}+l_{2}+m_{2}+n_{2}, N} \\
& =N \delta_{k_{1}+l_{1}+m_{1}+n_{1}, N} \delta_{k_{2}+l_{2}+m_{2}+n_{2}, N} \\
& =N \delta_{\kappa+\lambda+\mu+\nu \mathbf{0}}
\end{align*}
$$

where the origin point is $\mathbf{0}=(0,0)$. Using this constraint on the summation indices we can simplify the phase in Eq. (6.28):

$$
\begin{equation*}
\omega\left[\frac{1}{2}(\langle\nu, \kappa+\lambda+\mu\rangle+\langle\mu, \kappa+\lambda\rangle+\langle\lambda, \kappa\rangle)\right]=\omega\left[\frac{1}{2}(\langle\nu, \mu\rangle+\langle\lambda, \kappa\rangle)\right] . \tag{6.30}
\end{equation*}
$$

We now have

$$
\begin{align*}
\Xi_{\alpha \beta \gamma \delta}=\frac{1}{N^{4}} \sum_{\kappa \lambda \mu \nu} \omega & {\left[\frac{1}{2}(\langle\nu, \mu\rangle+\langle\lambda, \kappa\rangle)\right] } \\
& \times \omega[\langle\kappa, \alpha\rangle+\langle\lambda, \beta\rangle+\langle\mu, \gamma\rangle+\langle\nu, \delta\rangle] \delta_{\kappa+\lambda+\mu+\nu, \mathbf{0}} \\
=\frac{1}{N^{4}} \sum_{\kappa \lambda \mu \nu} \omega & {\left[\langle\kappa, \alpha\rangle+\frac{1}{2}\langle\lambda, \kappa\rangle+\langle\lambda, \beta\rangle\right] }  \tag{6.31}\\
& \times \omega\left[\left(\langle\mu, \gamma\rangle+\frac{1}{2}\langle\nu, \mu\rangle+\langle\nu, \delta\rangle\right)\right] \delta_{\kappa+\lambda+\mu+\nu, \mathbf{0}}
\end{align*}
$$

Lets investigate the sum of the first $\omega$ factor $\sum_{\kappa \lambda} \omega\left[\langle\kappa, \alpha\rangle+\frac{1}{2}\langle\lambda, \kappa\rangle+\langle\lambda, \beta\rangle\right]$. There are $N^{4}$ terms but the summation over $\kappa$ will only have nonzero terms when $\frac{1}{2} \lambda-\alpha=\mathbf{0}$. By the same reasoning, the sum over $\lambda$ leads to $\frac{1}{2} \kappa+\beta=\mathbf{0}$. $\alpha$ and $\beta$ are fixed so when $\lambda=2 \alpha$ or $\kappa=-2 \beta$ and we have $N^{2}$ surviving terms in the sum. Using these rules,

$$
\begin{align*}
\sum_{\kappa \lambda} \omega\left[\langle\kappa, \alpha\rangle+\frac{1}{2}\langle\lambda, \kappa\rangle+\langle\lambda, \beta\rangle\right] & =\sum_{\lambda} \omega[\langle\lambda, \beta\rangle] \sum_{\kappa} \omega\left[\langle\kappa, \alpha\rangle+\frac{1}{2}\langle\lambda, \kappa\rangle\right] \\
& =\sum_{\lambda} \omega[\langle\lambda, \beta\rangle] \sum_{\kappa} \delta_{2 \alpha-\lambda, \mathbf{0}}  \tag{6.32}\\
& =N^{2} \sum_{\lambda} \omega[\langle\lambda, \beta\rangle] \delta_{2 \alpha-\lambda, \mathbf{0}} \\
& =N^{2} \omega[\langle\alpha, \beta\rangle] \delta_{2 \alpha, \lambda}
\end{align*}
$$

Performing the summation over $\lambda$ first we would get

$$
\begin{equation*}
\sum_{\kappa \lambda} \omega\left[\langle\kappa, \alpha\rangle+\frac{1}{2}\langle\lambda, \kappa\rangle+\langle\lambda, \beta\rangle\right]=N^{2} \omega[2\langle\alpha, \beta\rangle] \delta_{2 \beta,-\kappa} \tag{6.33}
\end{equation*}
$$

### 6.4 The Four-point Structure Function $\Xi_{\alpha \beta \gamma \delta}$

Similarly, by switching which summation occurs first in the second $\omega$ factor, we have

$$
\begin{align*}
\sum_{\mu \nu} \omega\left[\left(\langle\mu, \gamma\rangle+\frac{1}{2}\langle\nu, \mu\rangle+\langle\nu, \delta\rangle\right)\right] & =N^{2} \omega[2\langle\gamma, \delta\rangle] \delta_{2 \gamma, \nu}  \tag{6.34}\\
& =N^{2} \omega[2\langle\gamma, \delta\rangle] \delta_{2 \delta,-\mu} \tag{6.35}
\end{align*}
$$

This gives us four equal expressions for $\Xi_{\alpha \beta \gamma \delta}$ :

$$
\begin{align*}
\Xi_{\alpha \beta \gamma \delta} & =\omega[2(\langle\alpha, \beta\rangle+\langle\gamma, \delta\rangle)] \delta_{2 \alpha, \lambda} \delta_{2 \gamma, \nu} \delta_{\kappa+\lambda+\mu+\nu, \mathbf{0}} \\
& =\omega[2(\langle\alpha, \beta\rangle+\langle\gamma, \delta\rangle)] \delta_{2 \alpha, \lambda} \delta_{2 \delta,-\mu} \delta_{\kappa+\lambda+\mu+\nu, \mathbf{0}}  \tag{6.36}\\
& =\omega[2(\langle\alpha, \beta\rangle+\langle\gamma, \delta\rangle)] \delta_{2 \beta,-\kappa} \delta_{2 \gamma, \nu} \delta_{\kappa+\lambda+\mu+\nu, \mathbf{0}} \\
& =\omega[2(\langle\alpha, \beta\rangle+\langle\gamma, \delta\rangle)] \delta_{2 \beta,-\kappa} \delta_{2 \delta,-\mu} \delta_{\kappa+\lambda+\mu+\nu, \mathbf{0}}
\end{align*}
$$

Plugging the first two Kronecker deltas in each line into the third, we have

$$
\begin{align*}
\Xi_{\alpha \beta \gamma \delta} & =\omega[2(\langle\alpha, \beta\rangle+\langle\gamma, \delta\rangle)] \delta_{\kappa+2 \alpha+\mu+2 \gamma, \mathbf{0}} \\
& =\omega[2(\langle\alpha, \beta\rangle+\langle\gamma, \delta\rangle)] \delta_{\kappa+2 \alpha-2 \delta+\nu, \mathbf{0}}  \tag{6.37}\\
& =\omega[2(\langle\alpha, \beta\rangle+\langle\gamma, \delta\rangle)] \delta_{-2 \beta+\lambda+\mu+2 \gamma, \mathbf{0}} \\
& =\omega[2(\langle\alpha, \beta\rangle+\langle\gamma, \delta\rangle)] \delta_{-2 \beta+\lambda-2 \delta+\nu, \mathbf{0}}
\end{align*}
$$

These four equations are equivalent to the requirement that

$$
\begin{equation*}
\Xi_{\alpha \beta \gamma \delta}=\omega[2\langle\alpha, \beta\rangle+2\langle\gamma, \delta\rangle] \delta_{\alpha-\beta+\gamma-\delta, \mathbf{0}} \tag{6.38}
\end{equation*}
$$

We put this in a more symmetric form by noting that if $\alpha-\beta+\gamma-\delta=\mathbf{0}$, then
$\langle\alpha+\gamma, \beta+\delta\rangle=\mathbf{0}$, which we now put to use:

$$
\begin{align*}
\langle\alpha, \beta\rangle+\langle\gamma, \delta\rangle & =\langle\alpha, \beta\rangle+\langle\gamma, \delta\rangle+\langle\beta, \gamma\rangle+\langle\gamma, \beta\rangle+\langle\delta, \alpha\rangle+\langle\alpha, \delta\rangle \\
& =\langle\alpha+\gamma, \beta+\delta\rangle+\langle\beta, \gamma\rangle+\langle\delta, \alpha\rangle  \tag{6.39}\\
& =\langle\beta, \gamma\rangle+\langle\delta, \alpha\rangle
\end{align*}
$$

Therefore, we have the pleasing expression

$$
\begin{equation*}
\Xi_{\alpha \beta \gamma \delta}=\omega[\langle\alpha, \beta\rangle+\langle\beta, \gamma\rangle+\langle\gamma, \delta\rangle+\langle\delta, \alpha\rangle] \delta_{\alpha-\beta+\gamma-\delta, \mathbf{0}} \tag{6.40}
\end{equation*}
$$

which can be further reduced to

$$
\begin{equation*}
\Xi_{\alpha \beta \gamma \delta}=\omega[\langle\alpha-\gamma, \beta-\delta\rangle] \delta_{\alpha-\beta+\gamma-\delta, \mathbf{0}} . \tag{6.41}
\end{equation*}
$$

There are five important properties of the four-point structure function $\Xi_{\alpha \beta \gamma \delta}$ to point out:
i.) $\alpha-\beta+\gamma-\delta=0$ constrains the four phase space points to label the vertices of a parallelogram in a cyclical fashion. This can be seen by showing opposite sides correspond to the same translation vector: $\alpha-\beta+\gamma-\delta=0 \rightarrow \beta-\alpha=$ $\gamma-\delta$ and $\gamma-\beta=\delta-\alpha$.
ii.) The quantity $\langle\alpha-\gamma, \beta-\delta\rangle$ that determines the phase of $\Xi_{\alpha \beta \gamma \delta}$ is just twice the area of the parallelogram formed by those four points.
iii.) $\Xi_{\alpha \beta \gamma \delta}=\Xi_{\beta \gamma \delta \alpha}$. This property is obvious from the trace representation. In phase space, it is the parallelogram constraint $\alpha-\beta+\gamma-\delta=0$ that enables the cyclic permutation of phase space points.
iv.) $\Xi_{\alpha \beta \gamma \delta}=\bar{\Xi}_{\gamma \beta \alpha \delta}=\bar{\Xi}_{\delta \gamma \beta \alpha}$. Swapping indices two positions away, which is the
same as reversing the index order, induces a complex conjugation of the original function. Making a calculation with a parallelogram traced in one direction gives the complex conjugate of the calculation made by cycling the other way. The oriented area swaps sign in the exponent inducing the complex conjugation.
v.) $\sum_{\beta \delta} \frac{1}{N^{2}} \Xi_{\alpha \beta \gamma \delta} \Xi_{\beta \sigma \delta \tau}=\delta_{\alpha \sigma} \delta_{\gamma \tau}$. This contractive property is proved below in Eq. (6.58).

It was interesting to discover that similar properties had been noted in the study of continuous $\star$-products $[144,33]$.

- Section 6.5


## Condition For Legitimate Transition Probabilities

Just as for the case of the transition rates for closed quantum systems, we might wonder by what rules must a set of transition rates abide. In this section, we investigate the relationship between the transition probabilities $P_{\alpha \gamma}$ and the Kraus coupling $\mathcal{B}_{\beta \delta}$ which leads to the criteria for legitimate sets of $P$ 's and $\mathcal{B}$ 's.

First we will invert the relationship between the transition rates and the Kraus coupling, writing the $\mathcal{B}$ 's in terms of the $P$ 's. We start with the following equation, which we derived from the Kraus form of a general, trace-preserving completely positive map.

$$
\begin{equation*}
P_{\alpha \gamma}=\sum_{\beta \delta} \mathcal{B}_{\beta \delta} \Xi_{\alpha \beta \gamma \delta} . \tag{6.42}
\end{equation*}
$$

Writing out $\Xi$ explicitly, we have

$$
\begin{equation*}
P_{\alpha \gamma}=\frac{1}{N} \sum_{\beta \delta} \mathcal{B}_{\beta \delta} \operatorname{Tr}\left(\hat{A}_{\alpha} \hat{A}_{\beta} \hat{A}_{\gamma} \hat{A}_{\delta}\right) \tag{6.43}
\end{equation*}
$$

In what follows we'll use two basic facts that apply to any $N \times N$ matrix $\hat{M}$ :

$$
\begin{align*}
& \sum_{\alpha} \hat{A}_{\alpha} \operatorname{Tr}\left(\hat{M} A_{\alpha}\right)=N \hat{M}  \tag{6.44}\\
& \sum_{\alpha} \hat{A}_{\alpha} \hat{M} \hat{A}_{\alpha}=N(\operatorname{Tr} \hat{M}) \mathbb{1} \tag{6.45}
\end{align*}
$$

Multiply Eq. (6.43) by $\hat{A}_{\gamma}$, sum over $\gamma$ and use Eq. (6.44). This gives us

$$
\begin{equation*}
\sum_{\gamma} P_{\alpha \gamma} \hat{A}_{\gamma}=\sum_{\beta \delta} \mathcal{B}_{\beta \delta} \hat{A}_{\delta} \hat{A}_{\alpha} \hat{A}_{\beta} \tag{6.46}
\end{equation*}
$$

Now multiply on the left by $\hat{A}_{\nu}$ and on the right by $\hat{A}_{\mu} \hat{A}_{\alpha}$, sum over $\alpha$ and use Eq. (6.45):

$$
\begin{align*}
\sum_{\alpha \gamma} P_{\alpha \gamma} \hat{A}_{\nu} \hat{A}_{\gamma} \hat{A}_{\mu} \hat{A}_{\alpha} & =\sum_{\beta \delta \alpha} \mathcal{B}_{\beta \delta} \hat{A}_{\nu} \hat{A}_{\delta} \hat{A}_{\alpha} \hat{A}_{\beta} \hat{A}_{\mu} \hat{A}_{\alpha}  \tag{6.47}\\
& =N \sum_{\beta \delta} \mathcal{B}_{\beta \delta} \hat{A}_{\nu} \hat{A}_{\delta} \operatorname{Tr}\left(\hat{A}_{\beta} \hat{A}_{\mu}\right) \tag{6.48}
\end{align*}
$$

Now take the trace of both sides:

$$
\begin{equation*}
\sum_{\alpha \gamma} P_{\alpha \gamma} \operatorname{Tr}\left(\hat{A}_{\nu} \hat{A}_{\gamma} \hat{A}_{\mu} \hat{A}_{\alpha}\right)=N \sum_{\beta \delta} \mathcal{B}_{\beta \delta} \operatorname{Tr}\left(\hat{A}_{\nu} \hat{A}_{\delta}\right) \operatorname{Tr}\left(\hat{A}_{\beta} \hat{A}_{\mu}\right)=N^{3} \mathcal{B}_{\mu \nu} \tag{6.49}
\end{equation*}
$$

where we've used the fact that $\operatorname{Tr}\left(\hat{A}_{\nu} \hat{A}_{\delta}\right)=N \delta_{\nu \delta}$. We thus have

$$
\begin{equation*}
\mathcal{B}_{\mu \nu}=\frac{1}{N^{2}} \sum_{\alpha \gamma} P_{\alpha \gamma} \Xi_{\mu \alpha \nu \gamma} \tag{6.50}
\end{equation*}
$$

This looks a lot like Eq. (6.42). The only essential difference is the prefactor of $1 / N^{2}$.
If someone gives us a proposed set of $P$ 's, how do we know whether that set describes an actual quantum process? If we were to imitate what we did for the $r$ 's
in the case of Hamiltonian evolution, our strategy for answering this question would be to use Eq. (6.50) to get $\mathcal{B}$ 's from those $P$ 's, and then to use Eq. (6.42) to get $P$ 's from those $\mathcal{B}$ 's. The final $P$ 's would have to be the same as what we started with. If not, those $P$ 's would not be legitimate. Let's try this strategy.

Let's start by rewriting Eq. (6.50) with different indices:

$$
\begin{equation*}
\mathcal{B}_{\beta \delta}=\frac{1}{N^{2}} \sum_{\sigma \tau} P_{\sigma \tau} \Xi_{\beta \sigma \delta \tau} \tag{6.51}
\end{equation*}
$$

Now insert this into Eq. (6.42):

$$
\begin{equation*}
P_{\alpha \gamma}=\sum_{\sigma \tau}\left(\frac{1}{N^{2}} \sum_{\beta \delta} \Xi_{\alpha \beta \gamma \delta} \Xi_{\beta \sigma \delta \tau}\right) P_{\sigma \tau} \tag{6.52}
\end{equation*}
$$

So our condition on the $P$ 's would be that $P$ should be an eigenvector of the matrix expressed by the quantity in parentheses, with eigenvalue 1. Here we think of $\alpha \gamma$ as one index of the matrix and $\sigma \tau$ as the other index. The matrix is an $N^{4} \times N^{4}$ elements. What is this matrix?

Here we show that the projector in question is just the identity matrix and disappointingly gives us no constraint:

$$
\begin{equation*}
\frac{1}{N^{2}} \sum_{\beta \delta} \Xi_{\alpha \beta \gamma \delta} \Xi_{\beta \sigma \delta \tau}=\delta_{\alpha, \sigma} \delta_{\gamma, \tau} \tag{6.53}
\end{equation*}
$$

Again, $\delta$ plays double duty as a Kronecker delta which has greek letter subscripts and phase space point delta which lives as a subscript. Due to the property of the trace, $\Xi$ is invariant under cycling of subscripts and can be expressed as

$$
\begin{equation*}
\Xi_{\alpha \beta \gamma \delta}=\omega[\langle\alpha-\gamma, \beta-\delta\rangle] \delta_{\alpha-\beta+\gamma-\delta, \mathbf{0}} \tag{6.54}
\end{equation*}
$$

Start by plugging Eq. (6.54) into both $\Xi$ 's in Eq. (6.53):

$$
\begin{align*}
& \frac{1}{N^{2}} \sum_{\beta \delta} \Xi_{\alpha \beta \gamma \delta} \Xi_{\beta \sigma \delta \tau} \\
&= \frac{1}{N^{2}} \sum_{\beta \delta} \omega[\langle\alpha-\gamma, \beta-\delta\rangle] \delta_{\alpha-\beta+\gamma-\delta, 0} \\
& \times \omega[\langle\tau-\sigma, \beta-\delta\rangle] \delta_{\tau-\beta+\sigma-\delta, 0}  \tag{6.55}\\
&= \frac{1}{N^{2}} \sum_{\beta \delta} \omega[\langle\delta, \alpha-\gamma+\tau-\sigma\rangle] \omega[\langle\beta, \gamma-\alpha+\sigma-\tau\rangle] \\
& \quad \times \delta_{\alpha-\beta+\gamma-\delta, 0} \delta_{\tau-\beta+\sigma-\delta, 0} .
\end{align*}
$$

Perform the sum over $\beta$ and choose the second Kronecker delta to update all previous values of $\beta$ in the surviving term:

$$
\begin{align*}
\frac{1}{N^{2}} \sum_{\beta \delta} \Xi_{\alpha \beta \gamma \delta} & \Xi_{\beta \sigma \delta \tau} \\
& =\frac{1}{N^{2}} \sum_{\delta} \omega[\langle\delta, \alpha-\gamma+\tau-\sigma\rangle] \omega[\langle\tau+\sigma-\delta, \gamma-\alpha+\sigma-\tau\rangle] \\
& \times \delta_{\alpha+\gamma, \tau+\sigma} \\
& =\frac{1}{N^{2}} \omega[\langle\tau+\sigma, \gamma-\alpha+\sigma-\tau\rangle] \sum_{\delta} \omega[2\langle\delta, \alpha-\gamma+\tau-\sigma\rangle]  \tag{6.56}\\
& \times \delta_{\alpha+\gamma, \tau+\sigma} \\
& =\frac{1}{N^{2}} \omega[\langle\alpha-\tau, \gamma-\sigma\rangle] \sum_{\delta} \omega[2\langle\delta, \alpha-\gamma+\tau-\sigma\rangle] \delta_{\alpha+\gamma, \tau+\sigma} \\
& =\omega[\langle\alpha-\tau, \gamma-\sigma\rangle] \delta_{\alpha-\gamma, \sigma-\tau} \delta_{\alpha+\gamma, \tau+\sigma} \\
& =\delta_{\alpha-\gamma, \sigma-\tau} \delta_{\alpha+\gamma, \tau+\sigma}
\end{align*}
$$

where we passed to the third to last line using the restriction of the Kronecker delta and passed to the second to last line using the fact $\sum_{\beta} \omega[\langle\beta, \alpha\rangle]=N^{2} \delta_{\alpha, 0}$. Rearrang-
ing the subscripts of the Kronecker deltas we have

$$
\begin{equation*}
\delta_{\alpha-\sigma, \gamma-\tau} \delta_{\alpha-\sigma,-(\gamma-\tau)} \tag{6.57}
\end{equation*}
$$

This indicates that the only way to have nonzero terms is if both $\alpha-\sigma$ and $\gamma-\tau$ are separately zero. We are left with the identity matrix with indices $\alpha \gamma$ and $\sigma \tau$ given by

$$
\begin{equation*}
\frac{1}{N^{2}} \sum_{\beta \delta} \Xi_{\alpha \beta \gamma \delta} \Xi_{\beta \sigma \delta \tau}=\delta_{\alpha, \sigma} \delta_{\gamma, \tau} . \tag{6.58}
\end{equation*}
$$

Unfortunately, this approach places no restriction whatsoever on the transition probabilities.

So where does the restriction we seek come from, if not from the above selfconsistency requirement? There must be a restriction beyond simple normalization, or else we could easily create illegal Wigner functions. Well, the $\mathcal{B}$ 's given by Eq. (6.50) (or Eq. (6.51)) must correspond to legitimate Kraus operators. Let's find the consequences of this requirement.

Recall the definition of $\mathcal{B}_{\beta \delta}$ :

$$
\begin{equation*}
\mathcal{B}_{\beta \delta}=\sum_{j} B_{j \beta} \bar{B}_{j \delta} \tag{6.59}
\end{equation*}
$$

where $B_{j \beta}$ is defined by

$$
\begin{equation*}
\hat{B}_{j}=\sum_{\beta} B_{j \beta} \hat{A}_{\beta} \tag{6.60}
\end{equation*}
$$

The Kraus condition, $\sum_{j} \hat{B}_{j}^{\dagger} \hat{B}_{j}=\mathbb{1}$, can therefore be expressed as

$$
\begin{equation*}
\sum_{\beta \delta} \hat{A}_{\delta} \mathcal{B}_{\beta \delta} \hat{A}_{\beta}=\sum_{j} \sum_{\beta \delta} \hat{A}_{\delta} \bar{B}_{j \delta} B_{j \beta} \hat{A}_{\beta}=\sum_{j} \hat{B}_{j}^{\dagger} \hat{B}_{j}=\mathbb{1} \tag{6.61}
\end{equation*}
$$

It is important that $\hat{A}_{\delta}$ be to the left of $\hat{A}_{\beta}$ in this condition. Now replace $\mathcal{B}_{\beta \delta}$ in this
condition by its expansion in terms of the $P$ 's, given by Eq. (6.51). This will give us a condition on the $P$ 's. The condition is

$$
\begin{align*}
\mathbb{1} & =\sum_{\beta \delta} \hat{A}_{\delta} \mathcal{B}_{\beta \delta} \hat{A}_{\beta} \\
& =\frac{1}{N^{2}} \sum_{\beta \delta \sigma \tau} \hat{A}_{\delta} P_{\sigma \tau} \Xi_{\beta \sigma \delta \tau} \hat{A}_{\beta} \\
& =\frac{1}{N^{3}} \sum_{\beta \delta \sigma \tau} \hat{A}_{\delta} P_{\sigma \tau} \operatorname{Tr}\left(\hat{A}_{\beta} \hat{A}_{\sigma} \hat{A}_{\delta} \hat{A}_{\tau}\right) \hat{A}_{\beta}  \tag{6.62}\\
& =\frac{1}{N^{2}} \sum_{\delta \sigma \tau} P_{\sigma \tau} \hat{A}_{\delta} \hat{A}_{\sigma} \hat{A}_{\delta} \hat{A}_{\tau} \text { [by Eq. (6.44) } \\
& =\frac{1}{N} \sum_{\sigma \tau} P_{\sigma \tau}\left(\operatorname{Tr} \hat{A}_{\sigma}\right) \hat{A}_{\tau}[\text { by Eq. }(6.45)] \\
& =\frac{1}{N} \sum_{\tau}\left(\sum_{\sigma} P_{\sigma \tau}\right) \hat{A}_{\tau} .
\end{align*}
$$

This condition will certainly be satisfied as long as

$$
\begin{equation*}
\sum_{\sigma} P_{\sigma \tau}=1 \tag{6.63}
\end{equation*}
$$

Moreover, the condition requires that $\sum_{\sigma} P_{\sigma \tau}=1$, as can be seen by multiplying both sides by $\hat{A}_{\rho}$ and taking the trace. So the condition expressed by Eq. (6.62) is equivalent to the standard normalization condition on the $P$ 's. We have still not found any significant restriction.

But we can obtain a non-trivial condition on the $P$ 's by looking again at the definition of $\mathcal{B}_{\beta \delta}$ :

$$
\begin{equation*}
\mathcal{B}_{\beta \delta}=\sum_{j} B_{j \beta} \bar{B}_{\mathrm{j} \delta} . \tag{6.64}
\end{equation*}
$$

If we regard $\mathcal{B}_{\beta \delta}$ as a matrix with indices $\beta$ and $\delta$, we first notice that the definition requires this matrix to be Hermitian. But this fact does not place any restriction on the $P$ 's, since Eq. (6.51) already tells us that $\mathcal{B}_{\beta \delta}$ is Hermitian regardless of the
values of the P's. More importantly, though, Eq. (6.64) tells us that $\mathcal{B}$ is a positive semidefinite matrix. That is, it has only non-negative eigenvalues. Every matrix of the form given in Eq. (6.64) is positive semidefinite, and every positive semidefinite matrix can be written in that form. (One can expand it in terms of its eigenvectors, for example.) So our condition on the $P$ 's is that the matrix

$$
\begin{equation*}
\sum_{\sigma \tau} P_{\sigma \tau} \Xi_{\beta \sigma \delta \tau} \tag{6.65}
\end{equation*}
$$

regarded as a matrix with indices $\beta$ and $\delta$, be positive semidefinite. We can express this condition by insisting that for any complex vector $v$ with entries $v_{\beta}$, we must have

$$
\begin{equation*}
\sum_{\sigma \tau \beta \delta} P_{\sigma \tau} \bar{v}_{\beta} \Xi_{\beta \sigma \delta \tau} v_{\delta} \geq 0 \tag{6.66}
\end{equation*}
$$

Here is yet another way of expressing the condition: for any matrix $\hat{M}$, we must have

$$
\begin{equation*}
\sum_{\sigma \tau} P_{\sigma \tau} \operatorname{Tr}\left(\hat{M}^{\dagger} \hat{A}_{\sigma} \hat{M} \hat{A}_{\tau}\right) \geq 0 \tag{6.67}
\end{equation*}
$$

(We get this from Eq. (6.66) by letting $\sum_{\beta} v_{\beta} \hat{A}_{\beta}=\hat{M}$.) We can get at least an interesting consequence of this condition by considering the special case in which $\hat{M}$ is a unitary transformation corresponding to an symplectic (area-preserving) affine transformation of the phase space. In that case, $\hat{M} \hat{A}_{\tau} \hat{M}^{\dagger}$ is equal to $\hat{A}_{\mu(\tau)}$, where $\mu(\tau)=S \tau+\delta, S$ being a representation of $S p[2, \mathbb{Z}(N)]$ and $\delta$ being some translation vector. With this kind of $\hat{M}$ in Eq. (6.67), we get the condition

$$
\begin{equation*}
\sum_{\tau} P_{\mu(\tau), \tau} \geq 0 \tag{6.68}
\end{equation*}
$$

for every such $\mu$.
The primary result presented in this section was the constraint that $\mathcal{B}$ must be
a positive-definite matrix in order for its corresponding transition probabilities to describe legitimate quantum dynamics. We also learned that while the transition rates may take negative values, they must satisfy the normalization constraint in Eq. (6.63) which requires conservation of quasiprobability.

- Section 6.6


## Lindblad Dynamics

It is interesting to explore the Markovian scenario where open quantum systems obey the Lindblad master equation

$$
\begin{equation*}
\dot{\rho}=\frac{1}{i \hbar}[\hat{H}, \hat{\rho}]+\mathcal{L}(\hat{\rho}) \tag{6.69}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{L}(\rho)=\sum_{j}\left(\hat{L}^{j} \hat{\rho} \hat{L}^{j^{\dagger}}-\frac{1}{2}\left\{\hat{L}^{j^{\dagger}} \hat{L}^{j}, \hat{\rho}\right\}\right) \tag{6.70}
\end{equation*}
$$

with the Lindblad dissipators $\hat{L}^{j}$ representing different interactions with the environment and the bracket $\{\cdot, \cdot\}$ is the anti-commutation bracket $[88,19]$. Using the same techniques as those used to derive the closed system Wigner current, we have

$$
\begin{equation*}
\frac{\partial W_{\alpha}}{\partial t}=\sum_{\gamma} \mathscr{J}_{\alpha \gamma}+\frac{1}{N} \operatorname{Tr}\left(\mathcal{L}(\hat{\rho}) \hat{A}_{\alpha}\right) \tag{6.71}
\end{equation*}
$$

We'll focus on a single dissipator that appears in the summation:

$$
\begin{align*}
& \frac{1}{N} \operatorname{Tr}\left[\left(\hat{L}^{j} \hat{\rho} \hat{L}^{j^{\dagger}}-\frac{1}{2}\left\{\hat{L}^{j^{\dagger}} \hat{L}^{j}, \hat{\rho}\right\}\right) \hat{A}_{\alpha}\right] \\
& =\frac{1}{N} \operatorname{Tr}\left[\left(\sum_{\delta} L_{\delta}^{j} \hat{A}_{\delta}\right)\left(\sum_{\gamma} W_{\gamma} \hat{A}_{\gamma}\right)\left(\sum_{\beta} \bar{L}_{\beta}^{j} \hat{A}_{\beta}^{\dagger}\right) \hat{A}_{\alpha}\right. \\
& \left.-\frac{1}{2}\left\{\left(\sum_{\delta} \bar{L}_{\delta}^{j} \hat{A}_{\delta}^{\dagger}\right)\left(\sum_{\beta} L_{\beta}^{j} \hat{A}_{\beta}\right),\left(\sum_{\gamma} W_{\gamma} \hat{A}_{\gamma}\right)\right\} \hat{A}_{\alpha}\right]  \tag{6.72}\\
& =\sum_{\gamma} W_{\gamma} \sum_{\delta \beta}\left[L_{\delta}^{j} \bar{L}_{\beta}^{j} \Xi_{\delta \gamma \beta \alpha}-\frac{1}{2} \bar{L}_{\delta}^{j} L_{\beta}^{j} \Xi_{\delta \beta \gamma \alpha}-\frac{1}{2} \bar{L}_{\delta}^{j} L_{\beta}^{j} \Xi_{\gamma \delta \beta \alpha}\right] \\
& =\sum_{\gamma} W_{\gamma} \sum_{\beta \delta}\left[L_{\beta}^{j} \bar{L}_{\delta}^{j}\left(\Xi_{\alpha \beta \gamma \delta}-\frac{1}{2} \Xi_{\alpha \delta \beta \gamma}-\frac{1}{2} \Xi_{\alpha \gamma \delta \beta}\right)\right] \\
& =\sum_{\gamma} \mathscr{K}_{\alpha \gamma}^{j}
\end{align*}
$$

where

$$
\begin{equation*}
\mathscr{K}_{\alpha \gamma}^{j} \equiv W_{\gamma} \sum_{\beta \delta}\left[L_{\beta}^{j} \bar{L}_{\delta}^{j}\left(\Xi_{\alpha \beta \gamma \delta}-\frac{1}{2} \Xi_{\alpha \delta \beta \gamma}-\frac{1}{2} \Xi_{\alpha \gamma \delta \beta}\right)\right] . \tag{6.73}
\end{equation*}
$$

$\mathscr{K}_{\alpha \gamma}^{j}$ is the discrete Wigner current due to bath interaction. As in the case for closed system dynamics, we have a decomposition of the current into the value of the Wigner function at the point of departure and the transition rate out of that point. We can write

$$
\begin{equation*}
\frac{\partial W_{\alpha}}{\partial t}=\sum_{\gamma}\left(\mathscr{J}_{\alpha \gamma}+\sum_{j} \mathscr{K}_{\alpha \gamma}^{j}\right)=\sum_{\gamma}\left(r_{\alpha \gamma}+\sum_{j} s_{\alpha \gamma}^{j}\right) W_{\gamma}, \tag{6.74}
\end{equation*}
$$

where we have the bath transition rate

$$
\begin{equation*}
s_{\alpha \gamma}^{j}=\sum_{\beta \delta}\left[L_{\beta}^{j} \bar{L}_{\delta}^{j}\left(\Xi_{\alpha \beta \gamma \delta}-\frac{1}{2} \bar{\Xi}_{\alpha \gamma \beta \delta}-\frac{1}{2} \Xi_{\alpha \gamma \delta \beta}\right)\right] . \tag{6.75}
\end{equation*}
$$

The Wigner function must take only real values so it is important to check that the bath transition rates will respect this rule. We have seen that swapping indices
of $\Xi_{\alpha \beta \gamma \delta}$ two locations away inverts the orientation of the parallelogram and gives a complex conjugate. In the sum $\sum_{\beta \delta} L_{\beta}^{j} \bar{L}_{\delta}^{j} \Xi_{\alpha \beta \gamma \delta}$, there will pairs of complex conjugate terms found when swapping $\beta$ and $\delta$ indices so it will have a real valued contribution. We treat the next two terms in tandem which are

$$
\begin{equation*}
-\frac{1}{2} \sum_{\beta \delta} L_{\beta}^{j} \bar{L}_{\delta}^{j}\left(\bar{\Xi}_{\alpha \gamma \beta \delta}+\Xi_{\alpha \gamma \delta \beta}\right) \tag{6.76}
\end{equation*}
$$

Note that the points $\alpha$ and $\gamma$ occur in sequence in the indexing so they determine a side of a parallelogram. Imagine some parallelogram with fixed points $\alpha \gamma \sigma \rho$.In the summation over $\beta$ and $\delta$, there will be a term when $\beta=\sigma$ and $\delta=\rho$ with a contribution $L_{\sigma}^{j} \bar{L}_{\rho}^{j} \bar{\Xi}_{\alpha \gamma \sigma \rho}$ (the other $\Xi_{\alpha \gamma \rho \sigma}$ doesn't cyclically define a parallelogram and has no contribution). For another term, the points will be swapped with a contribution $L_{\rho}^{j} \bar{L}_{\sigma}^{j} \Xi_{\alpha \gamma \sigma \rho}$. We see we have two terms that are complex conjugates, hence a real contribution. This argument can be made for any parallelogram therefore the Lindblad transition rates will always lead to real valued Wigner functions.

### 6.6.1. Application: Quantum Thermodynamics

There has been much development in the field of quantum thermodynamics and it is the focus of much ongoing research. It may be quite interesting and informative to understand Carnot engines [52], Otto cycles [81], or the smallest possible refrigerator [89] from a discrete phase space perspective. A study of the emergence of thermodynamic laws from quantum mechanics [80] may also be revealing in phase space. Here we give the simple application of the Lindblad current showing how quantum mechanical heat and work can be understood on discrete phase space following Schumacher et al. [118]

The expectation value for the energy of a system is $\langle E\rangle=\operatorname{Tr} \hat{\rho} \hat{H}$. Energy can change either due to internal changes of the state of the system or due to external
factors changes the system Hamiltonian:

$$
\begin{equation*}
\frac{d}{d t}\langle E\rangle=\operatorname{Tr} \dot{\hat{\rho}} \hat{H}+\operatorname{Tr} \hat{\rho} \dot{\hat{H}} \tag{6.77}
\end{equation*}
$$

These terms on the RHS correspond to thermodynamic heat and work, respectively. Let $\mathcal{P}_{Q}$ and $\mathcal{P}_{W}$ be the rates at which heat and work effect the system so that

$$
\begin{equation*}
\mathcal{P}_{Q}=\operatorname{Tr} \dot{\hat{\rho}} \hat{H} \quad \text { and } \quad \mathcal{P}_{W}=\operatorname{Tr} \hat{\rho} \dot{\hat{H}} \tag{6.78}
\end{equation*}
$$

Let us address both the heat and work terms individually in the phase space perspective. We start with the heat flow making our familiar expansions:

$$
\begin{align*}
\mathcal{P}_{Q} & =\operatorname{Tr} \dot{\hat{\rho}} \hat{H} \\
& =\operatorname{Tr}\left[\sum_{\alpha \beta} \dot{W}_{\alpha} \hat{A}_{\alpha} H_{\beta} \hat{A}_{\beta}\right] \\
& =N \sum_{\alpha} \dot{W}_{\alpha} H_{\alpha}  \tag{6.79}\\
& =N \sum_{\alpha \gamma}\left(\mathscr{J}_{\alpha \gamma}+\sum_{j} \mathscr{K}_{\alpha \gamma}^{j}\right) H_{\alpha} .
\end{align*}
$$

Notice that the heat contribution for the reversible closed system current disappears because

$$
\begin{align*}
\sum_{\alpha \gamma} \mathscr{J}_{\alpha \gamma} H_{\alpha} & =\sum_{\alpha \beta \gamma} \frac{1}{i \hbar} H_{\alpha} H_{\beta} W_{\gamma}\left(\Gamma_{\alpha \beta \gamma}-\Gamma_{\beta \alpha \gamma}\right) \\
& =\sum_{\alpha \beta \gamma} \frac{1}{i \hbar} W_{\gamma}\left(H_{\alpha} H_{\beta} \Gamma_{\alpha \beta \gamma}-H_{\alpha} H_{\beta} \Gamma_{\alpha \beta \gamma}\right)  \tag{6.80}\\
& =0
\end{align*}
$$

where we just swapped indices $\alpha$ and $\beta$ in the second term of the second line and
used the commutativity of phase space functions. We are left with

$$
\begin{equation*}
\mathcal{P}_{Q}=N \sum_{\alpha \gamma} \sum_{j} \mathscr{K}_{\alpha \gamma}^{j} H_{\alpha}=N \sum_{\alpha \gamma} \sum_{j} s_{\alpha \gamma}^{j} W_{\gamma} H_{\alpha} . \tag{6.81}
\end{equation*}
$$

This means that any heat flow in our system is due to the current contributions stemming from the Lindblad dissipators.

The expression for energy flow due to work has the phase space form

$$
\begin{align*}
\mathcal{P}_{W} & =\operatorname{Tr} \rho \dot{H} \\
& =\operatorname{Tr}\left[\sum_{\alpha \beta} W_{\alpha} \hat{A}_{\alpha} \dot{H}_{\beta} \hat{A}_{\beta}\right]  \tag{6.82}\\
& =N \sum_{\alpha} W_{\alpha} \dot{H}_{\alpha} .
\end{align*}
$$

As we would anticipate, the contributions to heat and work show up as an average weighted over phase space by the Wigner function. The other factor in the summation can be interpreted as the rate of heat and work energy transfer to the system at that phase space point.

## Section 6.7

## Summary

In this chapter, we have laid the foundation for open quantum system dynamics on discrete phase space. With the current interest in applying phase space methods to finite dimensional quantum systems combined with the necessity of understanding noisy processes of such systems for practical gain in quantum computing or sensing, this may provide important insights. In particular, we have formulated a discrete Wigner transition probability for quantum channels. As with the study of quantum states, negativity is a unique quantum feature now arising in transition probabilities.

A curious geometrical property of the four-point structure function was explored. We formulated the constraint that allows for verification that a set of transition probabilities belongs to a legitimate quantum process. Lastly, we introduced the discrete Wigner current corresponding to Markovian dynamics in phase space.

## Chapter 7

## Conclusion

In this thesis, we introduced new tools to analyze the dynamics of quantum systems in phase space and to differentiate classical and quantum behavior. The Wigner function which is the most well-known representation of continuous variable quantum systems in phase space played a central role in this effort. We understood the Wigner function as the unique function on phase space that produces the correct marginal probabilities for a quantum state. Furthermore, it was shown how it quantifies the symmetry of the wavefunction about a specific point in phase space. The mathematical structure was displayed in a foreshadowing of how a Wigner function on discrete phase space may be constructed.

Moving on to dynamics of continuous Wigner functions, we studied how the Wigner current vector field can inform efforts to stabilize significant negative regions in the Wigner function. Specifically, we extended the application of the Wigner current to Lindblad dynamics describing open quantum systems. Using a Gauss's law argument, we explained why a Hamiltonian must have anharmonic terms in order to combat the detrimental effects of the environments diffusion current. Additionally, we offered some thoughts pertaining to certain shapes of negative regions may be particularly advantageous when working to create sustained, or at least long-lived
negative regions.
Motivated by our study of dynamics from a current perspective in continuous systems, we investigated how this approach would translate to finite dimensional quantum systems. Restricting ourselves to dimensions of prime powers, we reviewed how one may construct a discrete Wigner function in complete analogy with the continuous version. All the relevant structure in the continuous case were reproduced including, the Fourier transform, displacement operators, symplectic transformations, and the phase point operators. The beautiful duality between the geometry of the space and the physical description.

Once we had laid the foundation and reviewed relevant the relevant development in the literature, we studied Hamiltonian dynamics on the discrete phase space. We derived a discrete Wigner current and its associated transition rates. We carefully studied the patterns that arose in the space of transition rates which quantify the transfer rate of quasiprobability from one point in phase space to another. It was emphasized that a sign of non-classical behavior occurs not only when the Wigner function takes negative values, but also when the transition rates become negative. Hamiltonians consisting of single displacement operators where shown to have a geometric structure in the space of transition rates which was completely predictable with knowledge of the powers of $\hat{X}$ and $\hat{Z}$. Interestingly, for such Hamiltonians, one can always represent it as a free particle up to a symplectic transformation. On the way, we understood how the transition rates explain the fact that quantum probabilities update with the second order in time. Finally, a constraint was derived that enables one to check whether a set of transition rates do indeed represent a legitimate reversible quantum process.

Building off the closed system case, we studied general quantum channels described by completely-positive trace-preserving maps. In phase space, these maps
correspond to a set of transition probabilities which we derived. Comparisons with classical dynamics reinforced the key idea that when transitioning from classical to quantum systems on phase space, negatively plays a distinguishing role. Our preconceptions about transition probabilities had to be relaxed to include negative values. A Lie theoretic understanding of the relationship between transition rates and transition probabilities was presented. As with the transition rates, we derived a constraint that allows one to identify sets of legitimate quantum transition probabilities. Interesting geometric facts were then displayed when we investigated the evaluation of the four-point structure function $\Xi_{\alpha \beta \gamma \delta}$. Lastly, we showed how to treat the Lindblad dynamics in the discrete case.

Now that we have described these basic tools that may be used to study quantum dynamics, there are a number of steps that may be taken in the future. In the case of continuous systems, a proof that one can or cannot stabilize significant negative regions without altering the natural Bosonic environment begs to be given. Advancements made in this area with the exploration of the effect of different nonlinear terms may be made in tandem with pioneering experimental work in which great parametric control over Hamiltonian terms is achieved [36].

Much effort has been put into erecting the formalism of phase space quantum mechanics so that it stands independently from its Hilbert space sibling [58, 97, 30]. We showed that transition probabilities describing reversible Hamiltonian dynamics must be described as $Z_{\alpha \gamma}=\frac{1}{N} \operatorname{Tr}\left[\hat{U}^{\dagger} \hat{A}_{\alpha} \hat{U} \hat{A}_{\gamma}\right]$ where $\hat{U}$ is some unitary operator enacting the Hilbert space dynamics. A nice step would be to construct such a proof without having to refer back to the Hilbert space as a crutch.

The structure functions are objects that also merit further investigation. Any equation that has a product of $n-1$ operators in Hilbert space will result in a phase space equation with a $n$-point structure function. Intuitively, it seems that the
areas of geometric objects that will be found exponentiated in such a function will continue the trend from triangle to parallelogram in a zig-zag pattern across phase space[144]. The interesting topological aspect of the discrete space is that this zig-zag structure will curl around on itself eventually. This leads to a question of whether there may be limits to some type of path-integral formulation in discrete phase space. The idea being that in Hilbert space, one finds large products of differing unitary propagators describing different paths. Mapping this in discrete phase space may result in a simplification of the operator products as higher orders of the $n$-point structure function curls around on itself and exponentiated areas are repeated.

We have mentioned a number of applications of discrete Wigner currents in quantum information and quantum computing at the beginning of chapter 4 . It may be revealing to look more deeply at these studies using our description of quantum processes with transition probabilities. Studying quantum algorithms or random walks and decoherence with discrete phase space transition probabilities seems particularly interesting.

In this vein, a study of qubit systems may be particularly useful in light of the current technological advances [7]. We have avoided discussing qubits due to the technical issues that we pointed out in chapter 4. However, it is well understood how to treat systems of qubits in discrete phase space [3] so it is up to us to weave it into our description using transition probabilities. A further obvious generalization would be to study systems with dimensions equal to composite numbers. The easiest place to start in this regard is with the fifteen dimensional systems; fifteen is the smallest composite number not of characteristic two.

In this thesis, we have explored the general concept of dynamics in phase space. The continuous Wigner function theory is quite mature and we pushed its dynamical description in terms of currents further in the realm of open quantum systems. By
comparison, the discrete Wigner function theory is still in an adolescent state. We have introduced a description of the quantum dynamics on the discrete phase space in terms of transition probabilities and rates. We hope these new tools will find use both in practical applications and in conceptualization of the differences between the quantum and the classical worlds.

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[^0]:    ${ }^{1}$ A footnote in the paper where Wigner introduced his function states that "this expression was found by L. Szilard and the present author some years ago for another purpose."

